THE PERFORMANCE OF OPTION PRICING MODELS ON
HEDGING EXOTIC OPTIONS

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First Draft: April 5, 2003

Comments are welcome

Abstract

This paper examines the empirical performance of various option pricing models. The models are tested in the same way market practitioners use them: the models are fitted to all the market traded liquid option prices and are recalibrated whenever the model is used to mark-to-market the option under consideration or to set up hedging portfolios. The test is based on their effectiveness of hedging exotic options. Since exotic options are only traded in the over-the-counter market, historical data are unavailable, and the traditional approach of comparing market prices with model prices can no longer be used. We propose a new methodology to overcome this difficulty: model performance is based on accuracy of hedging strategies. Using historical S&P 500 futures option prices we show that the so-called practitioner’s Black-Scholes model performs better relative to other alternative models for valuing barrier options, but worse for valuing compound options. Our results also indicate that model performance depends on the degree of path dependence of the option under consideration as noticed by Hull and Suo (2002).

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1. Introduction

In the last decades the financial markets have witnessed a remarkable growth in both volume and complexity of the contracts that are traded in the over-the-counter market. Banks and other financial institutions rely heavily on mathematical models for pricing and hedging those contracts. Although the Black-Scholes (1973) model is still widely used amongst practitioners for option pricing as well as hedging, a variety of empirical studies have shown that the model does not adequately describe the underlying asset price process. A key assumption of the Black-Scholes model is that the underlying asset price follows a geometric Brownian motion with constant volatility. However, the implied volatilities from the market prices of the options tend to vary across both strike prices and maturities. This phenomenon is usually referred to as volatility smile and volatility term structure (see e.g. Rubinstein (1994)). As a result, inadequate use of the Black-Scholes model can lead to significant pricing and hedging errors. This is what is termed as model risk arising from the use of inadequate models (Green and Figlewski (1999)).

To reduce the model specification error, researchers have proposed various alternative models that relax the unrealistic assumptions in the Black-Scholes model. These extended models can be categorized into two groups: One-factor models including the constant elasticity of variance model (Cox (1996)) and the deterministic volatility function model (Dupire (1994), Derman and Kani (1994) and Rubinstein (1994)); Multi-factor models including the jump diffusion model (Merton (1976), Bates (1991) etc.) and the stochastic volatility model (Hull and White (1987), Heston (1993), Scott (1987), Stein and Stein (1991) and Wiggins (1987), among others). While these models are more realistic, the model risks to which the market participants expose still exist. Since each model relaxes some assumptions of the Black-Scholes model, model risk arises because
exotic options will depend on the model chosen. It is, therefore, a very important empirical issue in finance to test whether this type of risks can be reduced by using a more complicated alternative model.

A number of empirical tests on the performances of option pricing models have been conducted in recent years including Bakshi, Cao and Chen (1997), Dumas, Fleming and Whaley (1998), and Bates (1996), among others. Not surprisingly, all these empirical tests show some evidences that alternative models perform better than the Black-Scholes formula, although relative performances of those models are different. Most of the works so far have been focusing on the model’s out-of-sample performance in the following way: Parameters of the model under consideration are estimated such that the model prices for some European options match those prices that are observed in the market (e.g., from market transactions or broker quotes) at a specific time. The resulting models are then used to price some other European or American options at a later time. These model prices are then compare with the prices observed from the market at this time.

However, for vanilla options, model specification is less important because they are actively traded in the market place, and a great deal of information on the way the instruments are priced at any given time is readily available from brokers and other sources (Hull and Suo (2002)). Market participants tend to calibrate their models in a way that the model can fit the market prices as close as possible, and re-calibrate them whenever they mark the option to markets, or rebalance their hedging portfolios. In other words, the way models are tested in the current literature is not exactly the way models

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3 Bakshi, Cao, and Chen (1997) conduct a much comprehensive empirical study on the pricing and hedging performance of various alternative models for S&P 500 index options. The models they test include the Black-Scholes model, the stochastic volatility model, the stochastic and jump model and the stochastic volatility and stochastic interest rate model. Bates (1996) has tested the performance of the Black-Scholes model, the deterministic volatility function model, and the stochastic volatility and jump model using currency options. Dumas, Fleming and Whaley (1995), on the other hand, focus on the performance of a few deterministic quadratic volatility models and the implied volatility model.
are used in practice. Moreover, mathematical models are primarily used to compute the
prices and hedging parameters for exotic options that are traded in the over-the-counter
market and the lack of historical data on exotic option prices makes the testing approach
of comparing model prices and market prices impossible.

The primary objective of this paper is to empirically test the performance of
various models that are currently used in practice for valuing and hedging exotic options
such as barrier options, compound options, and lookback options, etc. Exotic option
prices are much more sensitive to model misspecification, since market prices for exotic
options are not available and market participants can not calibrate their model in the way
when the model is used for valuing vanilla options. Furthermore, our empirical analysis
tries to fit the model under consideration to the cross-section of all observed liquid
options and then test the performance of the model on contemporaneously hedging exotic
options, which is different from the out-of-sample testing approach in existing literature.
The performance of a model along the time-series dimension is not necessarily the same
as that along the cross-sectional dimension. Most of all, we test models in the way the
practitioners use models, i.e. recalibrate models frequently to market data. In this way,
the model risk is correctly estimated. Therefore, our research will be of interest to
academics in finance, as well as to practitioners and regulators in investments and risk
management.

Difficulties arise when the tests are based on exotic options. The common
approach of comparing the model prices with the market prices is impossible, since no
historical exotic options data are available. To overcome this difficulty, we propose a
new methodology: the model parameters are estimated at time \( t \) and a replicating
portfolio is synthetically created from the market data including the liquid option prices

\[ \text{footnote:4 We mainly focus on barrier options and compound options in this paper.} \]
and the underlying asset prices. At the next time step, the model price is compared with the value of the model’s replicating portfolio, then the model is recalibrated and the replicating portfolio is rebalanced. This procedure continues until the maturities of the exotic options. If the model is specified correctly or if the model works well, one unit of the exotic option can be hedged by an offsetting position in the replicating portfolio, and the expectation and variance of the hedging errors should be very small. For this reason, the average of the hedging error can be used as an indicator of the performance of the model under consideration when it is used to price and hedge exotic options.

Our testing method is similar to that of Melino and Turnbull (1995) in some way. Melino and Turnbull (1995) examine the effects of the stochastic volatility upon the pricing and hedging of long-term foreign currency options. Since long-term foreign currency options are not actively traded and there is no data available, their test analysis is based on dynamic hedging errors, too. However, our method is different in that the model is recalibrated frequently and the performance is based on their effectiveness on hedging exotic options.

Green and Figlewski (1999) investigate the performance of the Black-Scholes model when it is recalibrated daily to historical data. On the other hand, we recalculate the model to current market data rather than historical data. Hull and Suo (2002) adopt a similar approach as ours; however, instead of using the market data they assume there is a “true” model that generates the “true” vanilla and exotic options data. Therefore, test can still be based on the comparison between the candidate model prices and the “true” observed prices.

Using S&P 500 futures options, we consider the performances of the traditional Black-Scholes model and three other major alternative models: the constant elasticity of variance (CEV) model, the stochastic volatility model, and the jump diffusion model.
Similar to Bakshi et al (1997), we employ two different types of hedging strategies to gauge the relative performance of different models: the minimum variance hedging strategy and the delta-vega neutral hedging strategy. The method can be easily adopted to test other models that are currently used in practice.

Our finding indicates that model recalibration does have some effects on model relative performances. The Black-Scholes model outperforms alternative models on hedging some barrier options in terms of absolute hedging errors. For hedging compound options, however, the jump diffusion model or the stochastic volatility model performs better than the Black-Scholes model in general. In addition, our results show that the model performances also depend on the degree of path dependence of the options, as Hull and Suo (2002) note. For some of the barrier options (e.g. the long-term barrier options), the performances of all models are poor. Although the hedging performances are not necessarily the same as the pricing performance, our results indicate that the traders’ common practice of recalibrating the Black-Scholes model for pricing and hedging less liquid and exotic options may work well for some types of options, however, it may not work well for other types of options.

The rest of the paper is organized as follows. In Section 2, we briefly review various option pricing models being tested in this paper. In Section 3, we discuss the estimation and testing methodologies. We also illustrate the theoretic hedging errors with a misspecified model, specifically the hedging errors when estimated model parameters are updated frequently. In Section 4, we describe the S&P 500 futures and futures option data. In Section 5, estimated results are presented and in-sample fit is also discussed. In Section 6, the empirical results are presented and analyzed. Section 7 concludes the paper.
2. Option Pricing Models and Exotic Options

2.1 Theoretical Option Pricing Models: A Brief Review

In addition to the Black-Scholes model, three alternative competing models are considered in this paper: the constant elasticity of variance (CEV) model, the stochastic volatility (SV) model, and the jump diffusion (JUMP) model. For convenience, the risk free interest rate denoted by $r$, is assumed to be constant over time. We also assume that the market is frictionless and there are no arbitrage opportunities.

(1). Black-Scholes Model

The Black-Scholes (1973) model assumes that the underlying asset price follows a geometric Brownian motion,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dw_t,$$  \hspace{1cm} (1)

where $S_t$ is the price of underlying asset at time $t$, $w_t$ is a standard Brownian motion, $\mu$ and $\sigma$ are the instantaneous expectation and the volatility of the underlying asset returns respectively, and assume to be constant.

Under risk neutral probability measure, the asset price follows the following process:

$$\frac{dS_t}{S_t} = (r - q) dt + \sigma dw_t,$$ \hspace{1cm} (2)

where $q$ is the dividend rate of the underlying asset, and is assumed to be constant. Under these assumptions, the market is complete, and the derivatives can be perfectly hedged by the underlying asset and a risk free investment. For any derivative written on the asset and paying $g(S_T)$ at maturity $T$, its price $f(S,t)$ at time $t$ satisfies the following partial differential equation:

$$\frac{\partial f}{\partial t} + (r - q)S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rf = 0,$$
In particular, for a European call option with strike $X$, the payoff at maturity $T$ is $(S_T - X)^+$, and its price $C(t, S_t, X, \sigma)$ at time $t$ can be calculated by solving the equation (3). It is given by:

$$C(t, S_t, X, \sigma) = S_t e^{-q(T-t)} N(d_1) - X e^{-r(T-t)} N(d_2),$$

where,

$$d_1 = \frac{\ln \left( \frac{S_t}{X} \right) + (r - q + \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}}, \quad d_2 = d_1 - \sigma \sqrt{T-t}. \quad (4)$$

Equation (4) implies that option prices and their volatilities have a one-to-one correspondence. As a result, for each option price, the implied volatility can be computed by solving for the volatility that equates the model price with the observed market price. Under the assumptions of the Black-Scholes model, volatilities should be the same for options on the same asset with different strikes and maturities. However, empirical findings show that the implied Black-Scholes volatilities vary systematically with strikes and maturities, a phenomenon usually referred to as the volatility smile. In equity market the implied volatilities for options with the same maturity usually decrease as the strikes increase, in other words, the Black-Scholes model under-prices deep-out-of-the-money put options and over-prices deep-out-of-the-money call options. This volatility pattern is particularly noticeable since the 1987 crash.

The volatility smile implies that the lognormal assumption of the underlying asset returns does not hold empirically. To capture these stylized facts observed in empirical studies, two major extensions are made to the Black-Scholes model in the literature: the first extension relaxes the assumption on the volatility. The second extension allows for jumps in the dynamic process of the underlying asset returns. The alternative models can
be either one of the extensions or combination of the extensions. In this paper, we propose to test whether the performance of any one of these extensions is improved over the Black-Scholes model on hedging exotic options, such as barrier options and compound options.

(2). Constantly Elasticity of Variance Model

The constant elasticity of variance model (hereafter the CEV model), which is originally developed by Cox (1975), simply assumes that the local volatility of the underlying asset depends on its price level. Specifically, under the risk neutral probability measure the stochastic process of the underlying asset is assumed as the follows:

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma S_t^{\frac{\alpha}{2}} dw_t,$$

where $w_t$ is a standard Brownian motion, $\sigma$ and $\alpha$ are constant parameters, and $\alpha$, known as the elasticity factor, is restricted to the interval $[0, 2)$. In the limiting case $\alpha = 2$, the CEV model reduces to the Black-Scholes model. The general CEV process also nest square root process ($\alpha = 1$) and absolute diffusion process ($\alpha = 0$) as special cases.

Under the CEV process, the instantaneous volatility of underlying asset return is equal to $\sigma S_t^{\frac{\alpha}{2}-1}$ and hence is an inverse function of the underlying asset price. Both empirical observations and economic rationale support the inverse relationship between underlying price and the volatility. Consequently, by incorporating the negative correlation between the underlying price changes and the volatility changes, the CEV model could better describe the actual stock price behavior than the Black-Scholes model, and this is confirmed by the empirical studies of MacBeth and Merville (1980), and Emanuel and Macbeth (1982), among others.

It can be easily shown that the market is still complete under the assumptions in this model, thus options can be perfectly hedged by continuously rebalancing a replicating
portfolio which consists of the underlying asset and a risk free asset. Using the same
argument as in the Black-Scholes setting, one can show that a partial differential equation
similar to equation (3) still holds in this model: Let \( f(S_t, t) \) be the price of an arbitrary
derivatives at time \( t \), it satisfies the following partial differential equation:

$$
\frac{\partial f}{\partial t} + (r - q)S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^\alpha \frac{\partial^2 f}{\partial S^2} - rf = 0.
$$

(6)

Cox (1996) derives a closed form solution for a European call option price, \( C(t, S_t, X, T, \sigma, \alpha) \),

$$
C(t, S_t, X, T, \sigma, \alpha) = S_t e^{-q(T-t)} \sum_{n=0}^{\infty} \frac{e^{-x} x^n G(n + 1 + \frac{1}{2 - \alpha}, kX^{2-\alpha})}{\Gamma(n+1)} 
$$

$$
- X e^{-r(T-t)} \sum_{n=0}^{\infty} \frac{e^{-x} x^n \frac{1}{2 - \alpha} G(n + 1, kX^{2-\alpha})}{\Gamma(n+1 + \frac{1}{2 - \alpha})},
$$

(7)

where

\[
k = \frac{2(r - q)}{\sigma^2 (2 - \alpha) [e^{(r-q)(2-\alpha)(T-t)} - 1]};
\]

\[
x = kS_t^{2-\alpha} e^{(r-q)(2-\alpha)(T-t)};
\]

\[
G(m, v) = [\Gamma(m)]^{-1} \int_{v}^{\infty} e^{-u} u^{m-1} du.
\]

Schröder (1989) shows that equation (7) can be expressed in terms of the non-central chi-square distributions:

$$
C(t, S_t, X, T, \sigma, \alpha) = S_t e^{-q} Q(2, y, 2 + 2/(2 - \alpha), 2x)
$$

$$
- X e^{-r} Q(2, y, 2 - 2/(2 - \alpha), 2x),
$$

(8)

where \( y = kX^{2-\alpha} \), and \( Q(z, v, k) \) is the complementary noncentral chi-square distribution function evaluated at \( z \), with \( v \) degrees of freedom and noncentral parameter \( k \).
To evaluate $Q(z,v,k)$, we use the simple and efficient algorithm suggested by Schröder (1989), and when $z$ or $k$ is large we use the approximation to noncentral chi-square distribution derived by Sankaran (1963).

(3). the Pure Jump Diffusion Model

Merton (1976) develops a pure jump process to model the movement of stock prices subject to occasional discontinuous breaks. The model assumes the process of the underlying asset price as follows:

$$\frac{dS_t}{S_t} = (\mu - \lambda^* k^*) dt + \sigma dw_t^* + J^* dQ^*, \quad \text{(9)}$$

where $\lambda^*$ is the annual frequency of jumps, $k^*$ is the average jump size measured as a proportional increase in the asset price, $J^*$ is the random percentage jump conditional on a jump occurring, and

$$\ln(1 + J^*) \sim N(\ln(1 + k^*) - \frac{1}{2} \delta^2, \delta^2);$$

$Q$ is a Poisson counter with intensity $\lambda^*$, i.e. $\text{Prob}(dQ^* = 1) = \lambda^* dt$; $dQ^*$ and $dw_t^*$ are assumed to be independent.

Under these assumptions, the instantaneous mean of the jump diffusion process is given by:

$$\frac{1}{dt} E\left(\frac{dS_t}{S_t}\right) = (\mu - \lambda^* k^*).$$

The instantaneous variance of the total return of the process is given by:

$$\frac{1}{dt} \text{var}\left(\frac{dS_t}{S_t}\right) = \sigma^2 + \lambda^* (k^{*2} + (1 + k^*)^2 (e^{\delta^2} - 1)).$$

In the jump diffusion model, the instantaneous unanticipated underlying return consists of two parts: the first part is due to the normal underlying asset price changes, and the second part is due to the abnormal underlying asset price changes. Accordingly,
the variance of the total return of the underlying asset has two components as well: the component of the normal time volatility and the component of jump volatility. If there is no jump, i.e. $\lambda = 0$, then this model reduces to the Black-Scholes model. Compared to the Black-Scholes model, the jump diffusion model attributes the skewness and excess kurtosis observed in the implied distribution of the underlying asset returns to the random jumps in the underlying asset returns: the skewness arises from the average jump size and the excess kurtosis arises from the magnitude and variability of the jump component. Therefore it could, to some extent, be more capable of capturing the empirical features of underlying equity returns. However, the jump effect in pricing option may be less pronounced as the maturity of the option increases, as the jump effect may be cancelled out in the long run.

Since there is a jump risk in this setting, the market is no longer complete and the Black-Scholes no arbitrage argument cannot be employed here. To price derivatives, one needs to make further assumptions to get the dynamics of the diffusion process of the underlying returns under the risk neutral probability measure.

Bates (1991) shows that if the process for optimally invested wealth follow a geometric constant volatility jump-diffusion process and the representative consumer has time-separable power utility $^5$, the jump diffusion process under the risk neutral probability measure can be written as:

$$\frac{dS_t}{S_t} = (r - q - \lambda k)dt + \sigma dw_t + JdQ,$$

where $\lambda$ is the annual frequency of jumps, $k$ is the average jump size measured as a proportional increase in the asset price, $J$ is the random percentage jump conditional on a jump occurring, and

$^5$ For details of the assumptions, see Bates (1991) equations (8) and (9).
\[ \ln(1 + J) \sim N(\ln(1 + k) - \frac{1}{2} \delta^2, \delta^2), \]

\( Q \) is a Poisson counter with intensity \( \lambda \), i.e. \( \text{Prob}(dQ = 1) = \lambda dt \); \( dQ \) and \( dw \) are assumed to be independent.

If \( f(S_\tau, t) \) is the price of an arbitrary derivative at time \( t \), then using the risk-neutral argument, we can get the following partial differential equation:

\[ \frac{\partial f}{\partial t} + (r - q - \lambda k) S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \lambda E[ f(S(1 + J), t) - f(S_\tau, t) ] - rf = 0. \] (11)

For a European call option, its price \( C(S_\tau, X, t, \lambda, k, \delta) \) can be solved analytically (Merton (1976)):

\[ C(S_\tau, X, t, \lambda, k, \delta) = e^{-r(T-t)} \sum_{n=0}^{\infty} \frac{e^{-\lambda(T-t)}(\lambda(T-t))^n}{n!} (S_n e^{-\lambda T} N(d_{1n}) - X e^{-rT} N(d_{2n})) , \]

where,

\[ r_n = r - q - \lambda k + n \ln(1 + k)/(T-t) , \]

\[ d_{1n} = \frac{\ln(S_\tau / X) + (r - q + \frac{1}{2} (\sigma^2 (T-t) + n \delta^2))}{\sqrt{\sigma^2 (T-t) + n \delta^2}} , d_{2n} = d_{1n} - \sqrt{\sigma^2 (T-t) + n \delta^2} . \]

Unlike the Black-Scholes model and the CEV model, the jump diffusion model has two sources of uncertainty, and therefore is a multifactor model.

(4). the Stochastic Volatility Model (SV):

The stochastic volatility model was introduced by Hull and White (1987), Heston (1993), Scott (1987), Stein and Stein (1991) and Wiggins (1987), among others. In this type of models, volatility of the underlying asset price is assumed to be stochastic. As an example, we consider the case where volatility follows a mean-reverting Ornstein-
Uhlenbeck (OU hereafter) process, i.e. under the true probability, the asset price and the volatility follows the following processes:

\[
\frac{dS_t}{S_t} = \mu dt + \nu(t)dw_t^*,
\]

\[
d\nu(t) = k^* (\theta - \nu(t))dt + \sigma dz_t^*,
\]  

(12)

where \( k^*, \theta, \sigma \) are the speed of adjustment, long-run mean, and volatility of volatility parameters respectively. \( z_t^* \) and \( w_t^* \) are standard Brownian motions with a correlation coefficient \( \rho \).

The stochastic volatility model provides some additional flexibility over the Black-Scholes model to capture the empirical features found in the underlying asset returns. It attributes the skew effect to either the correlation between the underlying asset returns and the volatilities, or the volatility of volatility and attributes kurtosis effect to the volatility of volatility. However, the effects on option pricing may be small when time-to-maturity of the option is not long. This is because the volatility follows a continuous diffusion process and the ability that the volatility process generates enough short-run skewness or excess kurtosis is limited. Adding jumps in the process of underlying asset returns offers another flexibility to capture empirical features in the short run.

We choose the mean reverting OU volatility process instead of mean reverting square-root variance process of Heston (1993) or other processes for the following reasons: First, if the volatility follows a mean reverting OU process then the variance will follow the following process implied from Ito’s lemma:

\[
dV(t) = [\sigma^2 + 2k\theta \sqrt{V(t)} - 2kV(t)]dt + 2\sigma \sqrt{V(t)}dz_t^*.
\]  

(13)

When \( \theta \) is zero, equation (13) indicates that the variance follows a mean reverting square-root process assumed by Heston (1993).
Second, like Heston’s model, this model can allow for systematic volatility risk and is analytically tractable. Since volatility is not a tradable asset, options have to be priced in an incomplete market. Therefore we need to make some assumptions about the “price of volatility risk” to price options in the stochastic volatility framework. Other processes such as Hull and White (1987) assume the volatility risk is nonsystematic in order to generate analytically tractable models.

In this model, market price of volatility risk can be modeled as proportional to volatility. Under the risk neutral probability measure the underlying asset return and volatility processes are given as:

\[
\begin{align*}
\frac{dS_t}{S_t} &= (r - q)dt + \nu_t dw_t, \\
\nu_t &= k(\theta - \nu_t)dt + \sigma dz(t),
\end{align*}
\]

(14)

The neutralized volatility process has the same form as the true process with different speed of adjustment of the volatility process.

The price of a derivative \( f(S_t, \nu_t, t) \) at time \( t \) that pays \( g(S_T) \) at the maturity \( T \) satisfies the following partial differential equation:

\[
\frac{\partial f}{\partial t} + (r - q)S \frac{\partial f}{\partial S} + k(\theta - \nu_t) \frac{\partial f}{\partial \nu} + \frac{1}{2} \nu_t^2 S^2 \frac{\partial^2 f}{\partial \nu^2} + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial \nu^2} + \rho \sigma \nu_t S \frac{\partial^2 f}{\partial S \partial \nu} - rf = 0,
\]

subject to the boundary condition

\[
f(S_T, \nu_T, T) = g(S_T).
\]

(15)

Specifically, for a European call option written on the asset with strike price \( X \) and maturity \( T \), the price \( C(t, S_t, \nu_t, X) \) satisfies the differential equation subject to \( C(t, S_T, \nu_T, X) = \max \{S_T - X, 0\} \). A closed form solution can be expressed as (see Schöbe and Zhu (1999)):

\[
C(t, S_t, \nu_t, X) = S(t)e^{-q(T-t)}P_1 - Xe^{-r(T-t)}P_2.
\]

(16)

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Where,

\[ P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re}\left[ \frac{e^{-i\phi \ln \lambda}}{i\phi} f_j(t, T, S(t), \nu(t); \phi) \right] d\phi. \]

For \( j = 1, 2 \), \( f_j \) are the characteristic functions of \( P_j \) respectively, and are given as:

\[
f_1 = \exp \{ i\phi((r-q)(T-t) + \ln S(t) - \frac{1}{2}(1+i\phi)\rho \left[ \frac{\nu^2}{\sigma} + \sigma(T-t) \right] + \frac{1}{2} D(t, s_1, s_3) \nu^2 \\
+ B(t, s_1, s_2, s_3) \nu + C(t, s_1, s_2, s_3) \}, \tag{17}\]

where

\[
s_1 = -\frac{1}{2}(1+i\phi)(1-\rho^2) + \frac{1}{2}(1+i\phi)(1-\frac{2k\rho}{\sigma}),
\]

\[
s_2 = \frac{(1+i\phi)\theta \rho}{\sigma},
\]

\[
s_3 = \frac{1}{2}(1+i\phi)\frac{\rho}{\sigma}.
\]

\[
f_2 = \exp \{ i\phi((r-q)(T-t) + \ln S(t) - \frac{1}{2}i\phi \left[ \frac{\nu^2}{\sigma} + \sigma(T-t) \right] + \frac{1}{2} D(t, \hat{s}_1, \hat{s}_3) \nu^2 \\
+ B(t, \hat{s}_1, \hat{s}_2, \hat{s}_3) \nu + C(t, \hat{s}_1, \hat{s}_2, \hat{s}_3) \}, \tag{18}\]

where

\[
\hat{s}_1 = \frac{1}{2} \phi^2 (1-\rho^2) + \frac{1}{2}i\phi(1-\frac{2k\rho}{\sigma}),
\]

\[
\hat{s}_2 = \frac{i\phi \theta \rho}{\sigma},
\]

\[
\hat{s}_3 = \frac{1}{2} \frac{i\phi \rho}{\sigma}.
\]

The functions \( D, B, C \) in \( f_j \) are given by

\[
D(t, s_1, s_3) = \frac{1}{\sigma^2} (k - \gamma_1 \sinh(\gamma_1(T-t)) + \gamma_2 \cosh(\gamma_1(T-t))) \frac{\sinh(\gamma_1(T-t)) + \gamma_2 \cosh(\gamma_1(T-t))}{\cosh(\gamma_1(T-t)) + \gamma_2 \sinh(\gamma_1(T-t))},
\]
\[ B(t, s_1, s_3) = -\frac{1}{\sigma^2 \gamma_1} (k \theta \gamma_1 - \frac{(k \theta \gamma_1 - \gamma_2 \gamma_3) + \gamma_3 [\sinh(\gamma_1 (T-t)) + \gamma_2 \cosh(\gamma_1 (T-t))]}{\cosh(\gamma_1 (T-t)) + \gamma_2 \sinh(\gamma_1 (T-t))}) \]

\[ D(t, s_1, s_3) = -\frac{1}{2} \ln[\cosh(\gamma_1 (T-t)) + \gamma_2 \sinh(\gamma_1 (T-t))] + \frac{1}{2} k (T-t) \]

\[ -\frac{k^2 \theta^2 \gamma_1^2}{2 \sigma^2 \gamma_3^3} (\gamma_1 (T-t)) - \frac{\sinh(\gamma_1 (T-t))}{\cosh(\gamma_1 (T-t)) + \gamma_2 \sinh(\gamma_1 (T-t))} \]

\[ + \frac{(k \theta \gamma_1 - \gamma_2 \gamma_3) \gamma_3}{\sigma^2 \gamma_1^3} \cosh(\gamma_1 (T-t)) + \gamma_2 \sinh(\gamma_1 (T-t)) \] \]

with

\[ \gamma_1 = \sqrt{2 \sigma^2 s_1 + k^2}, \quad \gamma_2 = \frac{1}{\gamma_1} (k - 2 \sigma^2 s_3), \quad \gamma_3 = k^2 \theta - s_2 \sigma^2. \]

To evaluate equation (16), one needs to calculate the integrals in \( P_j \) through numerical methods. In this paper, Gaussian quadrature procedures in NAG are used, and these integrals can be evaluated efficiently and accurately for a broad range of reasonable parameters.

2.2 The Practitioner’s Models

Although researchers have made remarkable advances in developing much more realistic option pricing models, the most widely used valuation procedure among practitioners is, however, the simplest Black-Scholes model with ad hoc adjustments and recalibration. This so-called practitioner’s Black-Scholes (PBS hereafter) approach can be described as follows (Dumas et al (1998)): First, the Black-Scholes implied volatilities of all vanilla options are calculated at time \( t \). Second, implied volatilities are smoothed across strike prices and time-to-expiration to get a volatility surface at time \( t \). Third, other option prices at time \( t \) can be calculated from the Black-Scholes formula using volatility.
obtained corresponding point on the volatility surface. The above procedure is repeated whenever the model is used.

One of the key points of the PBS is to calibrate and recalibrate the Black-Scholes model to fit the cross-section European option prices exactly. In other words, the PBS is designed so that it always correctly prices all European options. This means the unconditional probability distribution of the underlying asset price at all future time is always correct. However, that does not mean that the joint distribution of the underlying asset price at different times is correct, too. Consequently, although the PBS can correctly prices a derivative whose payoff is contingent on the asset price at only one time, there is no guarantee that it can correctly price a derivative whose payoff is contingent on the underlying asset price at more than one time. The above points are made by Hull and Suo (2002). Exotic options, such as barrier options and compound options, are path-dependent options whose payoffs depend on the underlying asset price at different times may be mis-priced by the PBS. This is why we should look at exotic options when we test model risk in the way that is consistent with how these models are used in practice.

The option price models being tested in this paper are referred to as practitioner’s models because they are tested in the same way as practitioners use the model, i.e. recalibrate the model whenever it is used. The model parameters are obtained by fitting the model to the cross-section observed liquid option prices at time $t$ as close as possible, then other illiquid options (e.g. exotic options) can be priced and hedged at time $t$ by using the model with the estimated parameters.

Practitioner’s model implicitly assumes that the parameters are changing over time. The price of practitioner’s version of the model has the form of

$$C_t = f(t, S_t, \Theta_t, c_1, c_2, ..., c_n),$$
where \( C_t \) is the price of an option being priced and hedged, \( c_1, \ldots, c_n \) are cross-sectional observed vanilla option prices at time \( t \), \( \Theta_t \) is the vector of parameters at time \( t \). Interest rate is assumed to be constant.

A number of problems arise while implementing an option-pricing model in this way. As Buraschi and Corielli (2001) point out that the time-inconsistency can arise in three different ways: First, the dynamics of the underlying asset price obtained by fitting the model to a cross-section of observed option prices may be incompatible with the no-arbitrage evolution of the underlying asset price. Second, since only a finite set of option prices are observed at a specific date, the models have to be estimated based on these observed prices. Consequently, the estimators depend on the finite observed prices. The inferred option prices from the estimated model might not be consistent with the observed options. Third, updating the model from time to time implicitly assumes the fitted parameters can change over time. This implies that the model is internally inconsistent (Dybvig 1989) and may permit arbitrage in derivatives (Backus, Foresi, and Zin 1998). Although the model fits the observed liquid options almost exactly at any point, recalibration will generates hedging errors as we will show next.

2.3 Exotic Options

The option pricing model is evaluated based on its performance on valuing exotic options. Exotic options are options with more complicated payoffs than vanilla options. Most exotic options are traded in the over-the-counter market, and therefore there are no observed market prices available. As a result, exotic option price is more sensitive to the model specification than that for their vanilla counterparts. Two types of popular exotic options, barrier options and compound options, are chosen in our tests. However, our
testing approach is general; other options such as Asian options or lookback options can be used to test model performance as well.

A barrier option is an option where the payoff depends on whether the underlying asset’s price reaches a barrier level over the life of the option. There are many types of barrier options, such as knock-out options, knock-in options, and multiple barrier options. Without loss generality, we concentrate on valuing and hedging knock-out call options. A knock-out call option is an European call option that ceases to exist the first time the underlying hits a pre-specified barrier level, \( H \). Knock-out options can be classified as up-and-out (when \( S_t < H \)) options or down-and-out (\( S_t > H \)) options.

A compound option is an option that the underlying asset itself is an option. For example, a call on call compound option is a call option that the underlying asset is another call option. Suppose the underlying call option is a European option with maturity \( T_2 \) and strike \( X_2 \), and the price of the underlying call option at time \( T_1 \) can be denoted as \( C_1(T_1, S_{T_1}, X_2, T_2, T_1) \), then the price of the compound option \( C \) with maturity \( T_1 \) (\( T_2 > T_1 \)) and strike \( X_1 \) can be written as

\[
C(t, C_1, X_1, T_1, T_1 - t) = e^{-r(T_1 - t)} E^Q[(C_1 - X_1)^+] \\
= e^{-r(T_1 - t)} E^Q[(e^{-r(T_2 - T_1)} E^P (S_{T_1} - X_2)^+ - X_1)^+].
\]

Where \( P \) and \( Q \) are risk neutral probability measures at time \( T_1 \) and \( t \) respectively. There are three more main types of compound options: a call-on-put, a put-on-call, and a put-on-put. The idea of pricing these options is the same.

Pricing and hedging exotic options are much more complicated than their vanilla counterparts. Except for the models in the Black-Scholes Setting, there are no analytical European options pricing formula for compound and exotic options. As a result, the
A natural technique to use for valuing exotic options is Monte Carlo simulation. We will discuss the method using the SV model as an example. Hedging barrier options is hard since the delta of the options is discontinuous at the barrier. We will discuss the problem in Section 3.

We use the stochastic volatility model as an example to illustrate the simulation method of pricing barrier and compound European options.

First, divide $T-t$ (should be $T_i-t_i$ in the case of compound options) into $N$ intervals of length $\Delta t$, denote $t_i = t + i\Delta t$ and simulate the path followed by $S$ and $v$ as follows:

$$S(t_{i+1}) = S(t_i) + rS(t_i)\Delta t + \sqrt{v(t_i)S(t_i)}\epsilon_1(t_i)$$

$$v(t_{i+1}) = v(t_i) + k(\theta - v(t_i))\Delta t + \sigma \sqrt{v(t_i)S(t_i)}(\epsilon_1(t_i)\rho + \epsilon_2(t_i)\sqrt{1-\rho}).$$  \hspace{1cm} (19)

where $\epsilon_1(t_i)$ and $\epsilon_2(t_i)$ ($i = 0, 1, 2, \cdots, N$) are independent samples from a univariate standardized normal distribution;

Second, calculate the payoffs of the exotic option at maturity $T$. For a barrier option, the payoff is $P = \text{Max}(S_T - X)$, if $S_{t_i} < H$ for all $S_{t_i}$; and $P = 0$, otherwise. For a compound option, the payoff is $P = \text{Max}[C(T_1, S_{t_i}, X_2, T_2 - T_1) - X_1, 0]$, where $C(T_1, S_{t_i}, v, X_2, T_2 - T_1)$ is the price of the underlying European option calculated by equation (16);

Third, repeat the first and the second steps for $M$ times, then $M$ of $P$s are calculated. Denote the $j$th $P$ by $P_j$. The estimated value of the option given by the stochastic volatility model is

$$C_b = \frac{e^{-rT}}{M} \sum_{j=1}^{M} P_j$$ for a barrier option, \hspace{1cm} (20)

or

\hspace{1cm}
\[ C_{\text{com}} = \frac{e^{-r_T}}{M} \sum_{j=1}^{M} P_j \] for a compound option. (21)

To reduce the variance of the estimators, antithetic variable technique can be used while carrying the simulations.

Although both compound options and barrier options are path-dependent options, they are different in terms of degree of path dependence.\(^6\) Since practitioners’ model tries to fit the observed vanilla option prices, any derivatives whose payoffs depend on the underlying price only at one time should be priced almost correctly. However, it might not the case if the practitioners’ model is used for pricing path depend options. As pointed out by Hull and Suo (2002), the pricing and hedging errors should depend on the degree of path dependence. As a result, we expect that the pricing and hedging performances for compound options should be difference from that for barrier options.

3. **Research Methodology**

   In this section, we discuss how to estimate the option pricing model parameters and how to test their performances using option-hedging criteria.

3.1 **Estimation of Model Parameters**

   Estimation procedure is a very important issue in option price modeling. Without an appropriate estimation method, model risk still exists even if the model is correctly specified. Generally speaking, there are two different methods for parameter estimation for the given model: the first one is to apply econometric estimation methods (such as ML or GMM) to obtain the required estimates using historical data of the underlying security prices. One of the potential problems of this method, as noted by Bakshi et al. (1997), is

\(^6\) The degree of path dependence is defined as the number of times the asset price must be observed to calculated the payoff of the option at maturity (see Hull and Suo 2000).
its stringent requirement on historical data. Also, for some models, it is not possible to yield the parameter estimates for the risk-adjusted processes that are necessary for valuation purpose (e.g. the stochastic volatility model). The second method is to imply the model parameters from the observed option prices. Parameters implied from option prices seem to be better estimators, which are confirmed by a number of empirical studies.\(^7\) This is because all the current information affecting the underlying asset price process is incorporated in the market prices including the current expectation of future volatilities.

For the second estimation method, there are two different approaches: one is to estimate the parameters at each time \(t\) by the cross-sectional option data and then average them to get the estimators. Another one is to estimate the parameters by pooling the cross-sectional time series options data.

We use the implied parameter estimation method to estimate our model parameters. Since our test approach is to recalibrate the model frequently, we estimate the model at each time \(t\) using the cross-sectional data. As a result, a time series of parameters are recorded. Details of the method are described as follows.

For a model that depends on a set of parameters \(\Phi = (a_1, a_2, \cdots, a_n)\), let us write the price of vanilla options (call or put) with strike \(X\) and maturity time \(T\) as

\[
C(t, \Phi, S, X, T),
\]

where \(t\) and \(S\) represent the current time and stock price respectively. At each time \(t\), there are many vanilla options with different strikes and maturities traded in the market place, and the corresponding market prices can be observed as \(\tilde{C}(t, S, X, T)\). Parameter vector \(\Phi\) at time \(t\) is chosen to minimize the sum of the squared errors (SSE), i.e.

\[
SSE(t) = \text{Min}_{\Phi} \left( \sum_{i,j} [\tilde{C}(t, S, X_i, T_j) - C(t, \Phi, S, X_i, T_j)]^2 \right). \tag{22}
\]

\(^7\) See, for example, Latane and Rendleman (1976), Melino and Turnbull (1990), Bates (1996), among others.
As we mentioned earlier, estimation method is a critical issue in modeling. Different objective functions in model estimation might yield different estimation and performance results. The error function can raise some problems, e.g. it assigns more weight to relatively expensive options (in-the-money option and long time-to-maturity option) and less weight to options with low valuations. One can use other alternative objective functions, such as percentage SSE or implied volatility SSE, in estimation. We choose the above function to estimate model parameters simply because it is used in most empirical studies, and this makes our results comparable with the existing empirical results.

3.2 Hedging Errors in Practitioners’ Models

Without historical data available for exotic options, model performance on valuing exotic options in our approach is based on the replication errors of a hedging portfolio held until the expiration of the target option. This approach is not only can be used to test model performance on valuing options without historical data but also can show the model’s dynamic performance. To illustrate the idea behind this testing method, we discuss the hedging errors when the model is implemented in the way that is consistent with how practitioners use the model.

Since the hedging portfolio can only be rebalanced discretely in practice, option hedging errors may either arise from discrete adjustments to the hedge or from model misspecification. By mis-specification, we mean the model is either incorrect or is implemented inconsistently, or both. In this paper, we focus on the hedging errors from
model misspecification, since hedging errors from discrete adjustments is small relative to the misspecification hedging errors if we rebalance the portfolio frequently.8

(1). Hedging errors when the hedged option prices are observable in the market

To illustrate the hedging errors from model misspecification, we assume that price of the underlying asset follows a two-factor stochastic volatility model given in equation (12) (we will call this the true model). Instead of using this model, we use a mis-specified model to calculate the hedging parameters. Denote the true price and model price of an option by \( C(t,S_t) \) and \( \hat{C}(t,S_t,\Theta_t) \) respectively, where \( \Theta_t \) is the vector of model parameters estimated at time \( t \), then the true option price satisfies the partial differential equation (15).

In the stochastic volatility setting, the volatility risk cannot be hedged only by delta hedging. Another derivative written on the same underlying asset has to be included in the hedging portfolio to control for the volatility risk.

Let us first discuss the hedging errors if the practitioners use the mis-specified but frequently recalibrated Black-Scholes model to hedge a target option. In Black-Scholes setting, the replicated portfolio involves the underlying asset and risk free investment. Consequently, the hedging portfolio consists of the option, the underlying asset and the risk free investment. The amounts of the underlying asset and the risk free investment are chosen so that the portfolio is insensitive to the changes of underlying asset and self-financing. This is the so called delta neutral hedging.

To hedge a target option, the replicate portfolio at time \( t \) consists of \( \Delta_t \) units of underlying asset \( S_t \) and \( B_t \) units of the risk free investment, and the value of the hedging portfolio at time \( t \) is

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8 Galai (1980) shows discrete adjustment hedging error is small relative to misspecification hedging error in the Black-Schole’s model setting. Lelan (1985) show the hedging error can be small if the model is correctly specified and rebalance the hedging parameters frequently.
\[ \pi_t = -C(t, S_t) + \Delta_t S_t + B_t. \] 

The portfolio is delta neutral if \( \frac{\partial \pi_t}{\partial S_t} = 0 \), and self-financing if \( \pi_t = 0 \). It is easy to show that \( \Delta_t = \frac{\partial C}{\partial S_t} \) and \( B_t = C(t, S_t) - \frac{\partial C}{\partial S_t} S_t \).

The hedging error from \( t \) to \( t + dt \) (ignoring high order terms) is:

\[ d\pi_t = -dC(t, S_t) + \frac{\partial C}{\partial S_t} dS_t + \left( C(t, S_t) - \frac{\partial C}{\partial S_t} S_t \right) rdtdS_t \]

\[ = \left\{ \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S_t} dS_t + \frac{1}{2} \frac{\partial^2 C}{\partial S_t^2} S_t^2 \varepsilon_t^2 dt + \frac{\partial C}{\partial v} dv + \frac{1}{2} \frac{\partial^2 C}{\partial v^2} \sigma^2 \varepsilon_t^2 dt + \frac{\partial^2 C}{\partial S_t \partial v} \varepsilon_t \varepsilon_v S_t v \sigma dt \right\} \]

\[ + \frac{\partial C}{\partial S_t} dS_t + C(t, S_t) rdtdS_t - \frac{\partial C}{\partial S_t} r S_t dt \]

\[ = \left( \frac{\partial C}{\partial S_t} - \frac{\partial C}{\partial S_t} \right) (r S_t dt - dS_t) + \frac{\partial C}{\partial v} (k(\theta - v) dt - dv) \]

\[ + \frac{1}{2} \frac{\partial^2 C}{\partial S_t^2} S_t^2 \varepsilon_t^2 dt + \frac{1}{2} \frac{\partial^2 C}{\partial v^2} \sigma^2 (1 - \varepsilon_t^2) dt. \]  

Equation (25) shows that the hedging errors in this case arise from the incorrect model hedging parameters and the time discretization as well. The first line of equation (25) arises from the mis-specified model used for hedging, and it is not zero even if the hedge portfolio is rebalanced continuously. Also, from this term we can see that the hedging errors can not only be from the misspecification of the volatility but also from the
misspecification of the drift term of the underlying asset process if the instantaneous expectation of underlying asset returns under the real probability measure is not equal to the risk free rate. This part of the hedging errors can be decomposed into two kinds of errors: errors from incorrect model and errors from incorrect implementation:

\[
\sum_{t=0}^{T} \left( \frac{\partial}{\partial S_t} C - \frac{\partial}{\partial S_t} \overline{C}(\Theta) \right) (rS_t dt - dS_t) + \sum_{t=0}^{T} \frac{\partial}{\partial \nu} (k(\theta - \nu) dt - d\nu)
\]

\[
= \sum_{t=0}^{T} \left( \frac{\partial}{\partial S_t} C - \frac{\partial}{\partial S_t} \overline{C}(\Theta) \right) (rS_t dt - dS_t) + \sum_{t=0}^{T} \frac{\partial}{\partial \nu} (k(\theta - \nu) dt - d\nu)
\]

\[
+ \sum_{t=0}^{T} \left( \frac{\partial}{\partial S_t} \overline{C}(\Theta) - \frac{\partial}{\partial S_t} \overline{C} \right) (rS_t dt - dS_t),
\]

where \(\overline{C}(\Theta)\) denotes the model price evaluated at the vector of parameters \(\Theta\). \(\Theta\) is constant through time.

The first two terms of equation (26) are hedging errors from misspecification if the model is implemented in the way that is consistent with the model assumptions. The second term is the error from inconsistency, i.e. the model is recalibrated frequently. The most interesting question is the interaction of these two kinds of hedging errors.

The second line of equation (25) arises from the discrete rebalance of the hedge portfolio. If the portfolio is rebalanced continuously, this part will be zero in the sense that the expected value and variance of it equal zero.

Next, we discuss the hedging errors if the practitioners use the frequently recalibrated stochastic volatility model to hedge the target option (e.g. an exotic option). Because the model used to hedge is a stochastic volatility model and it is well-known that the market is incomplete, options (vanilla option) written on the same underlying asset should be included in the hedging portfolio in order to hedge the volatility risk. The amount of the underlying, the bond and the hedging vanilla option should be chosen such
that the portfolio is not only insensitive to the changes of underlying asset prices and self-financing but also insensitive to volatility change. This is the so called delta-vega neutral hedging strategy.

Suppose that an option trader writes one unit of an exotic option \( C(t, S_t) \) (e.g. barrier option, compound option, or Asian options etc.) at time \( t \). At the same time he/she longs \( a_t \) units of European option \( C^E(t, S_t) \), \( b_t \) units of the underlying asset \( S_t \) and \( B_t \) units in the risk free investment to hedge the risk. The value of the portfolio the writer holds at the time \( t \) is

\[
\pi_t = -C(t, S_t) + a_t C^E(t, S_t) + b_t S_t + B_t. \tag{27}
\]

This portfolio is self-financing, delta and vega neutral if

\[
\pi_t = 0,
\]

\[
\frac{\partial \pi_t}{\partial S_t} = -\frac{\partial C}{\partial S_t} + a_t \frac{\partial C^E}{\partial S_t} + b_t = 0,
\]

\[
\frac{\partial \pi_t}{\partial \nu_t} = -\frac{\partial C}{\partial \nu_t} + a_t \frac{\partial C^E}{\partial \nu_t} = 0.
\]

The hedge parameters according to the specified model are, therefore,

\[
a_t = \frac{C_v}{C^E_v}, \tag{28}
\]

\[
b_t = \overline{C}_S - \frac{C_v}{C^E_v} C^E_S, \tag{29}
\]

\[
B_t = \overline{C} - a_t C^E - b_t S_t, \tag{30}
\]

where the notation \( \overline{C}_v \) denotes the derivative of \( \overline{C} \) with respect to \( \nu \), similar for \( \overline{C}_S, \overline{C}^E_S \) and \( \overline{C}^E_v \).

Using the same argument as for delta hedging, the hedging error from \( t \) to \( t+dt \) (ignoring high order terms) can be derived as:
\[ d\pi_t = -dC(t, S_t) + \frac{C_t}{C_v}dC_v^E + \left( \frac{C_t}{C_v} - \frac{C_t^E}{C_v} \right) dS_t \]

\[ + \left( C(t, S_t) - \frac{C_t}{C_v} C_v^E - \left( \frac{C_t}{C_v} - \frac{C_t^E}{C_v} \right) S_t \right) rdt \]

\[ = \left( C_t - C_v \right) (rS dt - dS_t) - \frac{C_t}{C_v} \left( C_v^E - C_v^E \right) rS dt - dS_t \]

\[ + C_t (k(\theta - \nu) dt - dv) - \frac{C_t}{C_v} C_v^E (k(\theta - \nu) dt - dv) \]

\[ + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} S_t^2 \nu^2 (1 - \nu_1^2) dt - \frac{C_t}{C_v} \frac{1}{2} \frac{\partial^2 C}{\partial S^2} S_t^2 \nu^2 (1 - \nu_1^2) dt \]

\[ + \frac{1}{2} \frac{\partial^2 C}{\partial \nu^2} \sigma^2 (1 - \nu_2^2) dt - \frac{C_t}{C_v} \frac{1}{2} \frac{\partial^2 C}{\partial \nu^2} \sigma^2 (1 - \nu_2^2) dt \]

\[ + \frac{\partial^2 C}{\partial S \partial \nu} S_t \nu \sigma (\rho - \nu_1 \nu_2) dt - \frac{C_t}{C_v} \frac{\partial^2 C}{\partial S \partial \nu} S_t \nu \sigma (\rho - \nu_1 \nu_2) dt . \quad (31) \]

Similar to the case above, the hedging errors both arise from model misspecification and time discretization. The first two lines of equation (31) are the hedging errors when the incorrect delta and vega are used for the calculations of the hedging parameters. The last three lines are the hedging errors arising from discrete adjustments of the hedging portfolio.

(2). Hedging errors when the hedged option is not traded in the market

If the target option trades in the over-the-counter market, its market prices are usually not available. Consequently, practitioners have to rely on models to price and hedge the option. For simplicity, we discuss the hedging errors in the one factor model setting.

Assume that the true volatility of the underlying asset price is \( \sigma \), and it is specified as \( \overline{\sigma} \) in the Black-Scholes model at time \( t \). As we have discussed earlier, the
delta hedging portfolio consists of \( \Delta, = \frac{\partial C}{\partial S_t} \) units of \( S_t \) and \( B_t = \overline{C}(t, \overline{\sigma}_t) - \Delta, S_t \) units in risk free investment at time \( t \). The value of the portfolio that the writer holds at time \( t \) in this case can be written as

\[
\pi_t = -\overline{C}(t, \overline{\sigma}_t) + \frac{\partial \overline{C}}{\partial S_t} S_t + (\overline{C}(t, \overline{\sigma}_t) - \frac{\partial \overline{C}}{\partial S_t} S_t).
\]

The target option price is a model price \( \overline{C}(t, \sigma_t) \) and not a real market price in equation (32) because a real market price is not available. Nevertheless, such a hedging strategy is still useful in identifying models that can set up more accurate hedges for the target option. The hedging error (ignoring high order terms) from time \( t \) to time \( t + dt \) is given by

\[
d\pi_t = -d \overline{C}(t, \overline{\sigma}_t) + \frac{\partial \overline{C}}{\partial S_t} dS_t + (\overline{C}(t, \overline{\sigma}_t) - \frac{\partial \overline{C}}{\partial S_t} S_t) r dt + \left( \overline{C}(t + dt, \overline{\sigma}_t) - \overline{C}(t + dt, \overline{\sigma}_{t+dt}) \right).
\]

\[
= \frac{1}{2} \frac{\partial^2 \overline{C}}{\partial S_t^2} S_t^2 (\overline{\sigma}_t^2 - \sigma_t^2 \varepsilon^2) dt + \left( \overline{C}(t + dt, \overline{\sigma}_t) - \overline{C}(t + dt, \overline{\sigma}_{t+dt}) \right)
\]

The first term in Equation (33) arises from the misspecification and discretization. The expectation and variance of this term are zero only if the model is correctly specified. Furthermore, the size of this term is proportional to the model’s gamma hedge parameter. As a result, exotic options and vanilla options may have different sensitivities to model misspecification. Intuitively, this is because the payoff of exotic products depends not only on the final price of the underlying asset but also on its prices throughout the life of the option. The second term in equation (33) arises from the deviation of model prices at time \( t \) due to model parameters updating.

Comparing the hedging errors in this case with the one we considered earlier gives us some ideas of the differences between the hedging errors of the two strategies. Adopting a similar approach used in Dalai (1983), we can rewrite equation (33) in the following form.
\[
d\pi_t = -dC(t, \sigma) + \frac{\partial C}{\partial S} dS_t + \left( C(t, \sigma) - \frac{\partial C}{\partial S} S_t \right) rdt
\]
\[
- dF_t - F_t rdt + \left[ C(t + dt, \sigma_t) - C(t + dt, \sigma_{t+dt}) \right],
\]
(34)

where \( F_t = C(t, \sigma) - \bar{C}(t, \sigma_t) \) and \( dF_t = F_{t+dt} - F_t \).

The first line of equation (34) is the same as considered in equation (24), while the second line contains the errors arising from the deviation between the model price and the true prices and as well as the deviation of the model prices from parameters updating.

Although the hedging strategies discussed above are applicable for any options, in practice there might be difficulties when hedging some of the exotic options. Consider, for example, an up-and-out barrier call option. The payoff of the option is given at maturity time \( T \) by

\[
(S_T - X)^+ \mathbb{1}_{\{\max_{t \leq T} S_t < \bar{H}\}},
\]

The delta and gamma of the option become large in absolutes value near expiration when the asset price is close to the barrier. A trader who adopts the delta hedging strategy would take large short (or long) positions in the underlying asset and make large adjustments to the hedging portfolio. To avoid such difficulties, the hedging positions are only rebalanced till 7 days to the maturity of the target option. There are also various other ways to hedge this type of options. One approach is known as static hedging strategy, which involves establishing a portfolio of vanilla options that approximately replicated the position of the target barrier option. The problem of the static hedging is that the vanilla options that needed to construct the replicating portfolio may not available in the market. Therefore, we do not investigate static hedging strategy in this paper.
3.3 Hedging Tests

Given the above discussions, we propose the following approach for testing the applicability of a model for valuing an exotic option:

(1) Estimate the model parameters under consideration using the method we have discussed in Section 3.1 at time $t$.

(2) Calculate the hedge parameters according to equations (28), (29), and (30) for delta-vega hedging. A hedging portfolio is constructed at time $t$.

(3) Suppose the hedging portfolio is rebalanced at intervals of length $\Delta$, i.e. the portfolio is rebalanced periodically at times $t + \Delta, t + 2\Delta, \cdots, t + m\Delta = T$. At time $t + \Delta$, re-estimate the model and calculate the hedging error of the portfolio between time $t$ and time $t + \Delta$, denoted by $\Delta \pi_1$

$$
\Delta \pi_1 = -C(t + \Delta, S_{t+\Delta}) + a_i C^E(t + \Delta, S_{t+\Delta}) + b_i S_{t+\Delta} + B_i (1 + r\Delta).
$$

(4) At time $t + \Delta$, reconstruct the hedging portfolio, and calculate the hedging error $\Delta \pi_2$ at time $t + 2\Delta$.

(5) Repeat steps from (1) to (4) until the maturity of the target option $T$, record the hedging errors $\Delta \pi_j$, for $j = 1, 2, \cdots, m$.

(6) Consider a set of target options indexed by $i$ with $i = 1, 2, \cdots, n$, repeat steps from (1) to (5), we will record $mn$ hedging errors totally, denoted as $\Delta \pi_{ij}$.

The total average dollar hedging error and absolute hedging error are calculated as:

$$
E(\Delta \pi) = \frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} \Delta \pi_{ij},
$$

$$
E(|\Delta \pi|) = \frac{1}{mn} \sum_{i=1}^{n} \sum_{j=1}^{m} |\Delta \pi_{ij}|.
$$

And the variances of the total average dollar hedging error and absolute hedging error are given as:
\[ \text{var}(\Delta \pi) = \frac{1}{mn-1} \sum_{i=1}^{n} \sum_{j=1}^{m} (\Delta \pi_{ij} - E(\Delta \pi))^2, \]

\[ \text{var}(|\Delta \pi|) = \frac{1}{mn-1} \sum_{i=1}^{n} \sum_{j=1}^{m} (|\Delta \pi_{ij}| - E(|\Delta \pi|))^2. \]

From our analysis the expectation and variance of the average hedging errors should be zero if the model is correctly specified. A model is said to perform better on hedging exotics than another model if the expectation and variance of the hedging error are less than the latter model.

4 Date Description

The dataset we use for our empirical analysis is the daily closing prices of S&P 500 index futures and futures options traded on the Chicago Mercantile Exchange (CME) from January 1993 to December 1993. This data set is chosen for the following reasons: first, indices are better representations of the economy than the arbitrary choice of individual stocks and they are the actively traded in the market; second, index options have been the focus of many previous studies in the literature (e.g. Bates (1991), Bakshi et al (1997), Dumas et al (1995)), and it makes our results comparable to those in the current literature.

There are four contract months for futures contracts: March, June, September, and December in each year. The last trading date of all the futures contracts are on the Thursday prior to the third Friday of contract months. The contract month of futures option could be any month in the year.

A futures option is in American style, i.e. a futures option may be exercised by the buyer on any business day that the option is traded before it expires. For options that expire in the March quarterly, the underlying asset is the futures contract for the month in which the option expires, and the last trading date is the same as the underlying futures
contracts. For options that expire in months other than those in the March quarterly cycle, the underlying futures contract is the next futures contract in the March quarterly cycle that is nearest the expiration of the option, and the last trading date is the third Friday of the contract month. On each trading day, there are many option prices with different strikes and maturities. Since the underlying futures and futures options are traded in the same market side by side, synchronization problems would be insignificant and can be ignored in our empirical analysis.

To discount cash flows, we use US T-bill rates as the proxy for the risk free interest rate. The daily T-bill middle rates for four maturities (one-, three-, six-, and 12-month) are obtained from the DataStream. The discount rates for periods other than the four maturities are calculated through interpolation.

The moneyness of an option is defined as

\[ m = \frac{F}{X} \]

where, \( F \) is the underlying futures price and \( X \) is the strike price of the option.

An option is said to be deep out-of-the-money if \( m < 0.90 \); out-of-the-money if \( 0.90 \leq m \leq 0.97 \); at-the-money if \( 0.97 < m < 1.03 \); in-the-money if \( 1.03 \leq m \leq 1.10 \); deep in-the-money if \( m > 1.10 \).\(^9\)

The time-to-maturity of an option is measured by the number of calendar days between the valuation and expiration dates. An option is classified as short-term if its days-to-expiration is less than 60 days; medium-term if it is between 60 and 180 days; long-term if it is more than 180 days.

There are totally 40100 options in the raw data, of which 17699 are call options. Maturities of all of the options are less than one year. To save computing time, only prices of call futures options in the data set are used. Moreover, some filters are applied to the

---

\(^9\) This is consistent with Bakshi et al (1997).
data: First, options with less than 7 days to expiration are excluded. These options have relatively small time premiums and their implied volatilities are extremely sensitive to liquidity-related biases. Second, options with moneyness $|m - 1|$ greater than ten percent are excluded. Deep in- and out-of-the-money options have small time premiums and contain little information about the volatility. Furthermore, they are infrequently traded, and their quotes are not updated frequently. Third, options that violates the arbitrage restrictions

$$F \geq C \geq \max[0, (F - X)]$$

are excluded. There are 13687 observations in the filtered data set, with an average of 54.7 contracts per trading day.

Table 1 reports the average prices and the number of observations in each category of the filtered data. Note that 42.4 percent of the 13687 observations are at-the-money options, 35.1 percent are out-of-the-money options and 22.4 percent are in-the-money options. Options with maturity days less than 180 days take up 78.7 percent of the total observations. The average call prices range from 0.12 for short-term out-of-the-money options to 42.69 for long-term in-the-money options.

The futures data set consists of 1012 futures contracts totally, and there are 8 different maturities every trading day. The longest futures are 1-year contracts.

Figure 3 shows the Black-Scholes implied volatility patterns with respect to moneyness and maturities. The implied volatilities are obtained by averaging of individual Black-Scholes call implied volatility within each moneyness-maturity category and across the days in the sample. Obviously, the futures options exhibit “volatility smile effect” across moneyness and maturities.
5 Model Parameter Estimation

Model parameters are implied from the observed option prices every day as described in Section 3.1. Since futures options traded in CME are in American style, we need to take into account the early exercise premium for American options. Although American options can be valued through some numerical methods, these methods are very computing expensive for the purpose of our estimation. For this reason, we use the quadratic approximation approach to deal with the early exercise premium for the American options.

5.1 Quadratic Approximation to the American Option Prices

For American options in Black-Scholes setting, the analytic approximation method was first suggested by MacMillan (1986) and then extended by Barone-Adesi and Whaley (1987). The American call option price, denoted by \( C(S,X,t) \), is given by

\[
C(S,X,t) = c(S,X,t) + A \left( \frac{S}{S^*} \right)^{\gamma} \quad \text{when } S < S^*,
\]

\[
C(S,X,t) = S - X \quad \text{when } S \geq S^*,
\]

where \( S^* \) is the critical price of the underlying futures above which the option should be exercised. This value can be calculated by solving the following equation:

\[
S^* - X = c(S^*,X,t) + \left\{ 1 - e^{-q(T-t)} N[d_1(S^*)] \right\} \frac{S^*}{\gamma}.
\]

And

\[
A = \left( \frac{S^*}{\gamma} \right) \left[ 1 - e^{-q(T-t)} N[d_1(S^*)] \right],
\]

\[
\gamma = -\left( \frac{2(r-q)}{\sigma^2} - 1 \right) + \sqrt{\left( \frac{2(r-q)}{\sigma^2} - 1 \right)^2 + \frac{8r}{\sigma^2(1-e^{-r(T-t)})}},
\]
where $N(d_1)$ is defined as before and $c$ is the corresponding European option price.

To evaluate (36), we use the algorithm proposed by Barone-Adesi and Whaley (1987). The iterative procedure converges very quickly and therefore American option prices in the Black-Scholes setting can be evaluated efficiently.

For the CEV model, however, no efficient analytic approximation method is available to value American options. In this paper, the early exercise premium in the Black-Scholes model with volatility $\sigma S_i^{0.5a-1}$ is used to approximate the early exercise premium for the CEV model. More specifically, the American option price for the CEV model is approximated by the corresponding European option price, which can be analytically calculated under the CEV model, plus the early exercise premium of the Black-Scholes model with volatility $\sigma S_i^{0.5a-1}$. We use this approximation because the early exercise premium is usually small for a futures option with maturity less than one year.

For the jump diffusion model, Bates (1991) derives an accurate and efficient quadratic approximation for valuing the American options. The approximation is an extension to the case in the Black-Scholes model: The American futures call option is approximated by:

$C(S, X, t) = c(S, X, t) + A \left( \frac{S}{S^*} \right)^{\gamma^*}$ when $S < S^*$, and

$C(S, X, t) = S - X$ when $S \geq S^*$, \hspace{1cm} (37)

where $S^*$ satisfies

$S^* - X = c(S^*, X, t) + \left[ 1 - \frac{\partial c(S^*, X, t)}{\partial S} \right] S^* / \gamma.$

$A = \left( \frac{S^*}{\gamma} \right) \left[ 1 - \frac{\partial c(S^*, X, t)}{\partial S} \right].$
and $\gamma$ is the positive root to the following nonlinear equation:

$$\frac{1}{2} \sigma^2 \gamma^2 + (-\lambda \mu - \frac{1}{2} \sigma^2)\gamma - r/(1 - e^{-r(T-t)}) + \lambda \left((1 + \mu)^r e^{0.5\gamma(\gamma-1)\delta^2} - 1\right) = 0.$$  

For the stochastic volatility model, we use the same constant volatility model to approximate American option prices, where the volatility is taken as the expected average volatility over the life of the option. Bates (1996) uses this approach to analyze the foreign exchange options in the jump diffusion model and the stochastic volatility, and shows that the approximation errors are negligible.

When we assume that the volatility process follows the mean reverting process given in equation (14), it can be shown (see Ball and Roma (1994) for details) that the expected average variance ($AV$) is given by

$$AV = E \left[ \frac{1}{T-t} \int_t^T v_s^2 d_s \right]$$

$$= \frac{\sigma^2}{2k} + \theta^2 + \frac{1}{T-t} (1 - e^{-k(T-t)}) \left( \frac{2\theta}{k} (v_t - \theta) - \frac{\sigma^2 - 2k(v_t - \theta)^2}{4k^2} (1 + e^{-k(T-t)}) \right)$$  \hspace{1cm} (38)

5.2 Parameter Estimation

As noted earlier, the valuation problem of futures options is equivalent to that of valuing options on a stocks or indices with dividend rate equals to the risk free interest rate. As a result, American and European futures options are evaluated for a given model by restricting the continuous dividend rate as the risk free rate.

Since it is quite slow when estimating the parameters for the stochastic volatility model, we further restrict the sample of futures option. On each trading day, we use all call options with moneyness greater than 0.94 and less than 1.03, maturity days less than 180 days as input to estimate that day’s spot volatility and relevant structural parameters.
SSE is obtained at the same time. The daily averages of the estimated parameters and SSEs for various models are reported in Table 2.

For the Black-Scholes model, the estimated volatility parameter over the sample period is 0.11 over the sample period. However, it varies considerably from day to day in the range of 0.09 to 0.15. Note that option prices and hedging ratios are very sensitive to the volatility parameter. Moreover, a constant volatility model usually generates quite different pricing and hedging results for exotic options from a non-constant volatility model. Figure 1 plots the changes in the estimated Black-Scholes volatility.

Figure 2 plots the time-paths of the two parameters of the CEV model. The sigma parameter in the CEV model ranges from 0.095 to 21.619 with the mean of approximately 6.332 and the standard deviation of 3.75. It varies through time significantly. The high standard deviation for the sigma parameter is generally expected since the variation of the elasticity factor has an exponential effect on that of sigma parameter; consequently a small deviation from the true value of the elasticity factor could lead to a relatively large deviation from the true value of the sigma parameter. The elasticity factor is relatively stable compared to the sigma parameter, even though it has a standard error of 0.251. The mean of the elasticity factor is 0.758, which indicates the negative correlation between the futures price changes and volatility changes.

The average of estimated volatility conditional on no jumps in the jump diffusion model is 0.074. It is not surprising that it is less than the implied Black-Scholes volatility because by allowing price jumps to occur, part of the total variance of the underlying asset return attributes to jumps. The model also attributes the negative skew and excess kurtosis to the jump risk, where jumps occur with a mean annual frequency of 1.592 times and a mean negative jump size of 0.219. The standard deviation of jump sizes conditional on a jump is 0.047. The changes if the parameters in the jump diffusion
model are plotted in Figure 3, which shows that the parameters are very volatile except for the volatility parameter.

For the stochastic volatility model, the mean of the estimated spot volatility is 0.111, which is slightly lower than that of Black-Scholes model. For the spot volatility process, the implied speed of adjustment of the volatility mean is 4.277, the long run mean volatility is 0.0591, and volatility of volatility mean is 0.349. These estimates strongly support the assumption of mean reverting stochastic volatility process. It also shows that the underlying price distribution has more leptokurtic than that of the lognormal distribution. The implied correlation coefficient between the underlying asset return and its volatility change is negative with a mean of –0.694. Figure 4 plots the changes in the parameters of the stochastic volatility model.

Overall, implied estimates of the models we have considered over the sample provide evidences for the parametric instability. Consequently, models with constant parameters fail to capture the evolution of the asset return distributions over time.

The estimation of the parameters for the three alternative models indicates that the underlying asset has an asymmetric return distribution, which is a feature that the Black-Scholes model fails to capture. The jump diffusion model and the stochastic volatility model offer further evidences of excess kurtosis of the return distribution. The improvement of the alternative models over the Black-Scholes model is further illustrated by the SSEs of the models: the SSE of the Black-Scholes is 18.48, which is the highest one among the models considered. The SSEs of the CEV, the jump diffusion and the stochastic volatility models are 7.77, 5.14, and 4.11 respectively. In other words, these alternative models give a much better in-sample fit than the Black-Scholes model, which is expected because they have more parameters, and therefore allow for more degrees of freedom. However, if some of the parameters are redundant and merely cause over
fittings of the data, the corresponding model will be penalized when out-of-sample performances or exotic option hedging performances are considered.

6 Hedging Performance on Exotic Options

In this section, we test the relative model performances on valuing exotic options using hedging performance as criteria. The analysis of Bakshi et al (1997), Dumas et al (1998) and other empirical studies in current literature are based on the out-of-sample model pricing and hedging performance. Our analysis different from those mainly in two respects: first models are recalibrated to the market option prices every trading day. Parameters of the models in each day are backed out by that day’s cross-sectional option prices, which is consistent with how models are used in practice. By doing this, the model parameters are allowed to change over time, and it can capture changes in the underlying asset return distribution that constant parameter models fail to capture. Second, our tests are based on the performances of the model on hedging exotic options at the same date when the model is fitted to the liquid option prices. Since exotic options are mainly trade in the over-the-counter market, some hedging criteria have to be used for our empirical analysis. The hedging effectiveness is measured by the total average hedging errors defined in Section 3.2.

We consider two different hedging strategies in this paper: minimum variance hedging strategy and delta-vega hedging strategy.

6.1 Minimum Variance Hedging Strategy

Minimum variance hedging strategy involves only the underlying futures as the hedging instrument. As noted by Ross (1995), the need for this kind of hedge arises in the contexts where a perfect delta-neutral hedge is not possible, either because of incomplete
market or because of model misspecifications and transactions costs. A minimum variance hedging portfolio is established in the following way: It consists of one unit of the hedged option and \( X \) units of the underlying futures, and the minimum variance hedging ratio \( X \) is determined by minimizing the variance of the hedging portfolio value.

To be more specific, suppose that an option trader writes one unit of option \( C \). If the writer relies on the minimum variance hedging strategy to hedge this option, then the value of the hedging portfolio at time \( t \) is:

\[
H = -C + X_s S + B,
\]

where \( B \) is the amount of risk free investment and \( B = C - X_s S \). The hedging portfolio is self-financing, and the change of \( H \) from \( t + dt \) to can be written as

\[
dH = -dC + X_s dS + Brdt.
\]

Its total variance of \( dH \) is given by

\[
\text{Var}(dH) = \text{Var}(dC) + X_s^2 \text{Var}(dS) - 2X_s \text{Cov}(dS,dC).
\]

By minimizing \( \text{Var}(dH) \), the hedging ratio can be found as:

\[
X_s = \frac{\text{Cov}(dS,dC)}{\text{Var}(dS)}.
\]

In the Black-Scholes or the CEV model, the market is complete, and the target option can be perfectly hedged by taking positions in the underlying asset and risk free investments. For these two models, the minimum variance hedging strategy is the same as the delta neutral hedging strategy, and the minimum variance hedging ratio is the delta of the hedged option.

However, in the jump diffusion or the stochastic volatility model, the market is no longer complete. Minimum variance hedging is not perfect for these cases in the sense that one cannot perfectly replicate the payoff of an option by only taking positions in the
underlying asset and risk free investments. In the stochastic volatility model, the minimum variance hedging ratio is given by

$$X_s = \frac{\partial C}{\partial S} + \rho \frac{\partial C}{\partial v} \frac{\sigma}{S_v}. \quad (39)$$

This shows that if the volatility is deterministic or stock returns are uncorrelated with volatility changes then minimum-variance hedging ratio is the same as delta hedging ratio. As we have mentioned in Section 5, $\rho$ is usually negative in equity markets, and as a result, minimum variance hedging ratio is generally different from the delta ratio to reflect the impact of volatility changes that are correlated with underlying asset returns. In fact, equation (39) shows that the minimum variance hedging ratio is usually less than the delta ratio.

Similarly, in the case of the jump diffusion model, the minimum variance hedging ratio is given by

$$X_s = \frac{\sigma^2}{\sigma^2 + V_j} \frac{\partial C}{\partial S} + \frac{\lambda}{S(\sigma^2 + V_j)} (E[J v(S(t,t + J))] - \mu C), \quad (40)$$

where $V_j = \lambda \mu^2 + \lambda (e^{\sigma^2} - 1)(1 + \mu^2)$.

The minimum variance hedging ratio in the jump diffusion model shows that if there is no jump risk (i.e. $\lambda = 0$) the minimum variance hedging is the same as the delta neutral hedging. However, if there is jump risk, the impact of jump risk is reflected in the minimum variance hedging ratio in the second term.

In this study, the target options to be hedged are barrier options and compound options on S&P 500 futures. To see the hedging effectiveness for options with different times-to-maturity, we consider three cases: 60, 90, and 180 days-to-expiration for the barrier or compound call option. We do not study options with longer time-to-maturity bases on the following considerations: the futures options we use to fit out model are less
liquid for maturity longer than 180 days. Moreover, our sample period is only one year, and there are 250 trading days in the year. The hedging procedure will be repeated each 7 days, as we will see next. The longer the term of the hedged option, the fewer hedging realizations we will get, and the less reliable might be the empirical results.

The hedging procedure is described as follows: On day $t$, model parameters are estimated by fitting futures option prices. The price of the exotic option under consideration $C_t$ can then be calculated from the model. To hedge this exotic option, a hedging portfolio is constructed with $X_s$ units of futures $F_t$, and $C_t$ units in risk free investment. Since the futures contracts require zero initial cash outlay, the total cost of such a portfolio is zero:

$$H_t = -C_t + X_s \cdot 0 + C_t = 0.$$ 

On day $t + dt$, the hedging portfolio is rebalanced. Using model parameters estimated on day $t+dt$, the value of the hedging portfolio is given by

$$H_{t+dt} = -C_{t+dt} + X_s (F_{t+dt} - F_t) + C_t (1 + r \frac{dt}{365}).$$

$H_{t+dt}$ is referred to as the hedging error on day $t + dt$. These steps are repeated until the option’s maturity date. This will give the average dollar hedging error, average absolute hedging error for this hedging realization as defined before.

This procedure tracks the hedging errors for one realization of the option being hedged. In order to perform empirical analysis, the procedure is repeated for every 7 days in the sample period and each repeat represents a realization of the sample path. A mean and a standard deviation are calculated for each tested model through these hedging errors, and the results are reported in Tables 3 and 4.

Table 3 reports hedging errors when the targeted options are up-and-out call options. The barrier level of each barrier option is set at 1.1 times of the initial futures
price, while strikes are set at 0.94, 0.96, 0.98, 1.00, 1.02, 1.04, and 1.06 times of initial futures prices, respectively. Options with three different time-to-maturities are considered: 60, 90 and 180 days-to-expiration. The hedging portfolio is rebalanced daily until 7 days to the maturity date.

Several observations can be made from the average absolute hedging errors in Table 3. First, with several exceptions, the Black-Scholes model outperforms other three alternative option pricing models. The CEV model is the second best performer. As for the jump and stochastic volatility models, the stochastic volatility model is better for hedging 60-day and 90-day barrier options but poorer for hedging 180-day options than the jump diffusion model. Second, for any given model, the hedging errors and standard deviations increase as maturities of barrier options increase. Note that the prices of barrier options decrease as the maturities increase; this result indicates hedging performance (relative to the option value) is quite poor for long-term barrier options. This confirms the result noted by Hull and Suo (2002) that model performance depend on the degree of path dependence of the option. For barrier options, the probability of the hitting the barrier is relatively large when the time-to-maturity is longer, and therefore the knockout feature becomes more important. To reconfirm this conclusion, we repeat the above hedging procedure for barrier options whose barriers are closer to the initial futures price (e.g. 1.05 times the futures price). The results are not reported here. The hedging errors, as we expected, are larger in this case than their counterparts in Table 3. Based on the sizes of dollar hedging errors, however, the results are confusing. No model seems to consistently perform better than another competing model. Based on the mean square errors, the observations from absolute errors still hold.

Table 4 reports hedging errors for compound options. The type of compound option considered is call on call option. The underlying call option is a futures option with

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60 days to expiration. The strike of the underlying call option is set as the initial futures price. Strikes of the call-on-call option are set as 1.5, 6.0, 10.5, 15.0, 19.5, 24.0, respectively. Like barrier options, three different days-to-expiration options are considered i.e. 60, 90 and 180 days-to-expiration options.

Based on absolute errors in Table 4, the CEV model performs the poorest. Both the jump diffusion model and the stochastic volatility model outperform the Black-Scholes model for the hedged options with 60 days to expiration. The jump diffusion model performs the best, followed by the stochastic volatility model. For long term (180 days) and lower $X_1$ compound options the Black-Scholes model outperforms alternative models. However, for long term high $X_1$ stochastic volatility model performs the best. All in all, Black-Scholes model is not recommended for hedging compound options especially for relative higher $X_1$.

Another observation from Table 4 is the hedging performances relative to the target option value for a given model do not change much while the times-to-expiration of the target options increase. Intuitively, increasing the time-to-expiration of a compound option does not affect the importance of exotic feature of it. As a result, the hedging performance of a compound option changes slightly while the time-to-expiration increases.

Based on the average dollar hedging errors, the sizes of hedging errors for the Black-Scholes model, the CEV model and the jump model are generally indistinguishable especially for low $X_1$, but in general the stochastic volatility model performs the best for short-term compound options. If the standard deviations of average errors are considered, the Black-Scholes model yields the smallest ones.

Comparing the performances of barrier options and compound options for any given model, we can see that the relative performance for short-term barrier options is better than for short-term compound options while the performance of long-term barrier
options is much poorer than long-term compound options. For Black-Scholes model, for instance, the average absolute hedging errors relative to the target option values are from 0.013 to 0.093 for 60-day barrier options, while they are from 0.061 to 0.126 for 60-day compound options. For 180-day barrier options the relative hedging errors are from 0.086 to 0.136 while they are from 0.055 to 0.094 for 180-day compound options.

6.2 Delta and Vega Neutral Hedging Strategy

As we mentioned in the last subsection, if there are more than one state variable in the model, the target exotic option cannot be perfectly hedged only by trading the underlying asset. For example, one more option underlying the same asset is needed to hedge the additional volatility risk in the stochastic volatility model. A hedging portfolio is called delta-vega neutral if the portfolio value is insensitive to changes in the underlying asset prices and volatilities.

For one-factor model, such as the Black-Scholes model and the CEV model, the delta neutral hedging is perfect, and adding one more instrument to the hedging portfolio is not necessarily based on the model settings. For the stochastic volatility model the delta-vega neutral hedging consists of the target exotic option, the underlying asset, another vanilla option and risk free investment. For the jump diffusion model, however, a perfect hedge is not possible.10 Similarly to Bakshi et al (1997) and Merton (1976), we only neutralize the diffusion risks but leave the jump risks uncontrolled in the jump diffusion model.

As in the minimum variance hedging, the target is a short position in a compound call on call option or a up and out barrier option with three kinds of days-to-maturity: 60 days, 90 days, and 180 days.

10 See Bates (1996), Bakshi et al (1997), and Merton (1976) etc.
The hedging procedure is similar to that for minimum variance hedging. The hedging positions are determined for each given model on day $t$ using implied parameters on the same day and current underlying futures price. On the next day, the hedging portfolio is revised and hedging errors are recorded. To replicate the hedged option, we choose different moneyness European options with maturities of 30 days longer than the hedged option to form different delta-vega hedging portfolios. We do this to make the results more reliable. Since the liquid options are American style futures options in our study, we use the model prices of the European futures options to calculate the hedging parameters.

The delta-vega hedging results for up-and-out barrier option are reported in panel A of Table 5. The results in panel A of Table 5 for the Black-Scholes, the CEV model, and the jump diffusion model are the same as those reported in Table 3, the only exception is the stochastic volatility model, since the stochastic volatility model needs one more option to neutralize the volatility risk. The general findings in minimum variance hedging strategy are still there. Another observation is that delta-vega hedging strategy leads larger hedging errors than minimum variance hedging strategy for the stochastic volatility model. This is in line with the finding of Melino and Turnbull (1995). Melino and Turnbull (1995) find that the introduction of an additional asset in the replicating portfolio causes a dramatic deterioration in the ability to replicate long-term options using short-term implicit parameters for a mis-specified model. They explain that the reason for this is because the error in calculating the vega value leads to incorrect amount investment in the European option.

As noted by Bakshi et al (1997), inclusion of another option in the delta-vega neutral hedge may not only neutralize the volatility risk but also reduce the remaining gamma risk in the hedge. To give each model a fair chance in the hedging performance
comparison, we follow Bakshi et al (1997) to implement the so-called delta-vega neutral hedge for the Black-Scholes model, the CEV model and the jump diffusion model, as well. Like for the stochastic volatility model, the so-called delta-vega hedging portfolio in the Black-Scholes, the CEV or the jump diffusion model consists of an additional option to neutralized vega of the hedging portfolio. The hedging results are reported in panel B of Table 5. The average absolute hedging errors show that the delta-vega hedging performance of Black-Scholes model outperforms all alternative models, followed by the CEV model, the stochastic model and the jump diffusion model.

The hedging results for compound options are reported in Table 6. Panel A of Table 6 reports the minimum hedging results for the Black-Scholes, the CEV and the jump diffusion models and the delta-vega hedging results for the stochastic volatility model. The average absolute hedging errors show the stochastic volatility model performs the best for low value of $X_1$. Based on the dollar hedging results, the stochastic volatility model performs poorer than the other three models for hedging short-term compound options, but better for long-term compound options. However, the reason for these results might be due to the fact that one more instrument is added in the delta-vega hedging portfolio for the stochastic volatility model. As Bakshi et al (1997) notice this instrumental option might not only neutralizes the volatility risk but also reduces the remaining gamma risk in the hedge. To see this, like for barrier options, we implement the delta-vega hedge strategy for the Black-Scholes model, the CEV model and the jump diffusion model, too. The results are reported in panel B of Table 6. Based on the average absolute hedging results, the Black-Scholes model significantly reduces its hedging errors, and performs the best.
7 Conclusions

Practitioners rely heavily on option pricing models to hedge and price financial derivatives. An inadequate model might cause mispricing of options and lead incorrect hedging strategies. Practitioners use a model differently from the way researchers originally derive it. Practitioners typically calibrated the model to the market for liquid options at least daily to ensure that the model stay close to the market, while researchers develop models with the assumption of constant parameters. Consequently, it is very important to test the effectiveness of a model currently being used in practice to reflect the way it is being used in the trading room.

In this paper, we test the relative performances of the Black-Scholes model and alternative option pricing models including the CEV model, the jump diffusion model and the stochastic volatility model. Our contributions to the current literature include: first, the models are tested in the way practitioners use them, i.e. the models are frequently recalibrated to cross-sectional liquid option prices and are tested their effectiveness of contemporaneously pricing and hedging other derivatives. Previous empirical works generally test the option pricing models with constant parameters and test the effectiveness of out-of-sample pricing and hedging liquid options. Second, the models are tested based on valuing exotic options. Since exotic options are traded in the over-the-counter market and their prices are not available, the general approach of comparing model prices and observed price cannot be used. We propose a new methodology for comparing different models that does not require data on option prices, i.e. the models are tested based on the accuracy of hedging strategies.

Our results show that although alternative models do improve their in-sample performance relative to the Black-Scholes model in terms of SSEs, they perform poorly
relative to the Black-Scholes model on hedging some exotic options (e.g. barrier options) as long as they are recalibrated frequently. As regarding hedging compound options, the Black-Scholes model, the CEV model and the jump model are distinguishable in terms of dollar hedging errors. The stochastic volatility model performs poorer than those three models for short-term compound options, but better for long-term compound options. For barrier options, the hedging errors are bigger for long-term options than for short-term options for any given models. For compound options, the hedging performances do not change much when the compound options go from short-term to long-term. The implication of these results is that the hedging performance of option pricing models depends on the degree of path dependence of the exotic options. For a given model, the hedging performances for short-term barrier options are better than that for short-term compound options; however for long-term options the hedging performances are poorer for barrier options than for compound options. This is because for short-term barrier options, the exotic feature is not important since the probability of hitting the barrier is small. However, for long-term barrier options, the exotic feature is much more important and the degree of path dependence is higher than that of compound options.

Although option pricing models are tested in the spirit of practitioner’s way in this paper, it is not the exact way practitioners implement the models. For example, Black-Scholes model is used as an interpolation tool to flatten the implied volatilities and recalibrated frequently in practice. We do this to make the performances are comparable across models, as Christoffersen and Jacobs (2001) note that the same objective function should be used while comparing model performances. Nevertheless, our paper is interesting in that it addresses the effect of current practice of implementing option pricing models among practitioners on model performances, and our results indicate that if the models are tested in a appropriate way, i.e. capturing the essence of the risk in the models
as they are used in practice, different relative model performances than those in current literature might be obtained.
References


Appendix A: Derivation of minimum variance hedging ratio for the stochastic volatility model and the jump diffusion model

In the stochastic volatility model, the stochastic processes of underlying asset and volatility under risk neutral probability measure are given by:

\[
\frac{dS_t}{S} = (r - q)dt + \nu(t)dw_t,
\]

\[
d\nu(t) = k[\theta - \nu(t)]dt + \sigma dz(t).
\]

Ito’s lemma implies:

\[
dC = \left( \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S} rS + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial \nu} k(\theta - \nu) + \frac{1}{2} \frac{\partial^2 C}{\partial \nu^2} \sigma^2 + \rho \frac{\partial^2 C}{\partial S \partial \nu} \right) dt
\]

\[
+ \frac{\partial C}{\partial S} \nu S dw(t) + \frac{\partial C}{\partial \nu} \sigma dz(t)
\]

Ignoring higher orders of \(dt\) and applying Ito’s lemma, we get:

\[
\text{var}(dS) = \nu^2 S^2 dt,
\]

\[
\text{cov}(dS, dC) = \left( \frac{\partial C}{\partial S} \nu^2 S^2 + \frac{\partial C}{\partial \nu} \rho \nu S \right) dt.
\]

The hedging ratio follows immediately.

In the jump diffusion model, the stochastic process of the underlying asset under risk neutral probability measure is given by

\[
\frac{dS_t}{S} = (r - q - \lambda k)dt + \sigma dw_t + JdQ.
\]

Ito’s lemma implies

\[
dC = \left( \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S} S(r - q - \lambda \mu) + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 \right) dt
\]

\[
+ \frac{\partial C}{\partial S} S \sigma dw(t) + [C(S(1 + J)) - C(S)] dq.
\]

The variance and covariance of the derivatives are given by
\[
\text{var}(dS) = S^2 \sigma^2 dt + S^2 \text{var}(Jdq)
\]

\[
= S^2 dt \left( \sigma^2 + \lambda \mu^2 + \lambda (e^{\delta s} - 1)(1 + \mu^2) \right) = S^2 dt (\sigma^2 + V_j)
\]

where \( V_j = \lambda \mu^2 + \lambda (e^{\delta s} - 1)(1 + \mu^2) \).

\[
\text{cov}(dS,dC) = E[dSdC]
\]

\[
= \frac{\partial C}{\partial S} \sigma^2 S^2 dt + \lambda Sdt E[J(C(S(1+J)) - JC(S)]
\]

\[
= \frac{\partial C}{\partial S} \sigma^2 S^2 dt + \lambda Sdt E[J(C(S(1+J))] - \lambda S \mu C dt.
\]

Consequently, in order to calculate, \( \text{cov}(dS,dC) \), one needs to evaluate \( E[J(C(S(1+J))] \). Unfortunately, if the target option is an exotic option, no analytic solution exists, and numerical method must be used to get the hedging ratio.
Table 1: Summary of the S&P 500 Futures Option Prices

<table>
<thead>
<tr>
<th>Moneyness $(m = F/X)$</th>
<th>Days-to-Expiration</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>&lt;60</td>
</tr>
<tr>
<td>0.90 ≤ m &lt; 0.94</td>
<td>0.1245</td>
</tr>
<tr>
<td></td>
<td>(411)</td>
</tr>
<tr>
<td>0.94 ≤ m &lt; 0.97</td>
<td>0.4864</td>
</tr>
<tr>
<td></td>
<td>(1191)</td>
</tr>
<tr>
<td>0.97 ≤ m &lt; 1.03</td>
<td>6.3830</td>
</tr>
<tr>
<td></td>
<td>(2423)</td>
</tr>
<tr>
<td>1.03 ≤ m &lt; 1.07</td>
<td>21.9188</td>
</tr>
<tr>
<td></td>
<td>(833)</td>
</tr>
<tr>
<td>1.07 ≤ m &lt; 1.10</td>
<td>36.7525</td>
</tr>
<tr>
<td></td>
<td>(260)</td>
</tr>
<tr>
<td>Subtotal</td>
<td>5118</td>
</tr>
</tbody>
</table>

Table 1 reports the average prices and number of observations in each category for futures call options from January 1993 to December 1993. There are about 250 trading days in 1993. The option is in the money if moneyness of the option is greater than 1.03; at the money if the moneyness is greater than 0.97 but less than 1.03; out of the money if the moneyness is less than 0.97. The moneyness of an option is defined as the ratio of the underlying futures price and the strike price of the option.
Table 2: Implied Parameters and SSEs

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Implied Volatility</th>
<th>SSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>BS</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.1134</td>
<td>0.1134</td>
</tr>
<tr>
<td></td>
<td>(0.010)</td>
<td>(0.010)</td>
</tr>
<tr>
<td>CEV</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_{CEV}$</td>
<td>6.3323</td>
<td>0.1152</td>
</tr>
<tr>
<td>$\alpha_{CEV}$</td>
<td>0.7577</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(3.752)</td>
<td>(0.010)</td>
</tr>
<tr>
<td></td>
<td>(0.251)</td>
<td>(12.266)</td>
</tr>
<tr>
<td>Jump Diffusion</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.0742</td>
<td>0.1373</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>1.5917</td>
<td></td>
</tr>
<tr>
<td>$\mu$</td>
<td>-0.2192</td>
<td></td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.0475</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.012)</td>
<td>(0.029)</td>
</tr>
<tr>
<td></td>
<td>(2.208)</td>
<td>(9.036)</td>
</tr>
<tr>
<td>SV</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k$</td>
<td>4.2766</td>
<td>0.1107</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.0591</td>
<td></td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.3486</td>
<td></td>
</tr>
<tr>
<td>$\rho$</td>
<td>-0.6936</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(6.302)</td>
<td>(0.175)</td>
</tr>
<tr>
<td></td>
<td>(0.131)</td>
<td>(0.146)</td>
</tr>
<tr>
<td></td>
<td>(0.175)</td>
<td>(0.146)</td>
</tr>
<tr>
<td></td>
<td>(0.020)</td>
<td>(8.120)</td>
</tr>
</tbody>
</table>

Note: The daily average of the estimated parameters for the Black-Scholes model (BS), the constant elasticity of variance model (CEV), the jump diffusion model (JUMP), and the stochastic volatility model (SV) using futures options data of 1993. Standard deviations are in parentheses. SSE is the daily average sum of squared errors.
Figure 1: The Time Path of Implied Black-Scholes Volatility

Figure 2: Time Paths of Parameters in the CEV Model

The time path of $\sigma$ in the CEV model

The time path of $\alpha$ in the CEV model.
Figure 3: Time Paths of Parameters of the Jump Diffusion Model

The time path of $\sigma$ in the jump model

The time path of $\lambda$ in the jump model

The time path of $\mu$ in jump model

The time path of $\delta$ in jump model
Figure 4: Time-Paths of the Parameters of the Stochastic Volatility Model

The time path of spot volatility

The time path of $k$ parameter

The time path of $\theta$ parameter

The time path of volatility volatility

The time path of $\rho$ parameter
Table 3: Minimum Variance Hedging Errors for Barrier Options

<table>
<thead>
<tr>
<th>Money-ness (X/F)</th>
<th>Days To Maturity</th>
<th>Absolute Hedging Errors</th>
<th>Dollar Hedging Errors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>BS</td>
<td>CEV</td>
</tr>
<tr>
<td>0.94</td>
<td>60</td>
<td>0.3393</td>
<td>0.3262</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.638)</td>
<td>(0.648)</td>
</tr>
<tr>
<td>0.96</td>
<td>90</td>
<td>0.2681</td>
<td>0.2798</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.500)</td>
<td>(0.514)</td>
</tr>
<tr>
<td>0.98</td>
<td>180</td>
<td>0.2337</td>
<td>0.2686</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.370)</td>
<td>(0.393)</td>
</tr>
<tr>
<td>1.00</td>
<td></td>
<td>0.2219</td>
<td>0.2721</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.274)</td>
<td>(0.299)</td>
</tr>
<tr>
<td>1.02</td>
<td></td>
<td>0.1810</td>
<td>0.2264</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.195)</td>
<td>(0.225)</td>
</tr>
<tr>
<td>1.04</td>
<td></td>
<td>0.1019</td>
<td>0.1358</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.117)</td>
<td>(0.148)</td>
</tr>
<tr>
<td>1.06</td>
<td></td>
<td>0.0344</td>
<td>0.0580</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.043)</td>
<td>(0.072)</td>
</tr>
<tr>
<td>0.94</td>
<td>90</td>
<td>0.7089</td>
<td>0.6723</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.949)</td>
<td>(0.954)</td>
</tr>
<tr>
<td>0.96</td>
<td>180</td>
<td>0.5447</td>
<td>0.5316</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.745)</td>
<td>(0.758)</td>
</tr>
<tr>
<td>0.98</td>
<td></td>
<td>0.3978</td>
<td>0.4185</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.544)</td>
<td>(0.568)</td>
</tr>
<tr>
<td>1.00</td>
<td></td>
<td>0.2831</td>
<td>0.3337</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.366)</td>
<td>(0.405)</td>
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<tr>
<td>1.02</td>
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<td>0.1870</td>
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<td>(0.224)</td>
<td>(0.278)</td>
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<td>1.04</td>
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<td>0.1564</td>
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<tr>
<td></td>
<td></td>
<td>(0.119)</td>
<td>(0.181)</td>
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<tr>
<td>1.06</td>
<td></td>
<td>0.0316</td>
<td>0.0737</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.040)</td>
<td>(0.101)</td>
</tr>
</tbody>
</table>

Note: Table 3 reports the minimum variance hedging errors for up and out barrier options. The strikes of the barrier option are set as 0.94, 0.96, 0.98, 1.00, 1.02, 1.04, 1.06 times of the underlying futures prices, and the barrier levels are set as 1.1 times of the underlying futures prices. Days-to-maturity are actual days. The hedge portfolios are rebalanced daily. The standard errors are given in parentheses.
<table>
<thead>
<tr>
<th>Strike Price ($X_t$)</th>
<th>Days To Maturity</th>
<th>Absolute Hedging Errors</th>
<th>Dollar Hedging Errors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>BS</td>
<td>CEV</td>
</tr>
<tr>
<td>1.5</td>
<td></td>
<td>0.6430</td>
<td>0.6737</td>
</tr>
<tr>
<td>6.0</td>
<td></td>
<td>(0.532)</td>
<td>(0.561)</td>
</tr>
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<td>10.5</td>
<td>60</td>
<td>0.5834</td>
<td>0.6102</td>
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<tr>
<td></td>
<td></td>
<td>(0.489)</td>
<td>(0.515)</td>
</tr>
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<td>15.0</td>
<td>60</td>
<td>0.4817</td>
<td>0.5028</td>
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<tr>
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<td>(0.429)</td>
<td>(0.451)</td>
</tr>
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<td>19.5</td>
<td>90</td>
<td>0.3712</td>
<td>0.3874</td>
</tr>
<tr>
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<td></td>
<td>(0.366)</td>
<td>(0.383)</td>
</tr>
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<td>24.0</td>
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<td>(0.302)</td>
<td>(0.316)</td>
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<td>(0.252)</td>
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<td>6.0</td>
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<td>(0.576)</td>
<td>(0.606)</td>
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<td>0.6700</td>
<td>0.7000</td>
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<td></td>
<td>(0.537)</td>
<td>(0.564)</td>
</tr>
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<td>(0.519)</td>
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<td>0.5117</td>
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<tr>
<td></td>
<td></td>
<td>(0.434)</td>
<td>(0.454)</td>
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<td>(0.348)</td>
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<td>(0.770)</td>
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<td>(0.702)</td>
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<td>(0.662)</td>
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<td></td>
<td></td>
<td>(0.598)</td>
<td>(0.620)</td>
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<td></td>
<td>0.5965</td>
<td>0.6167</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.557)</td>
<td>(0.579)</td>
</tr>
</tbody>
</table>

Note: Table 4 reports the minimum variance hedging errors for call on call options. The underlying call option is a futures option with 60 days-to-expiration. Strikes of call are set as 1.50, 6.0, 10.5, 15.0, 19.5, 24.0. Days-to-maturity are actual days. The hedge portfolios are rebalanced daily. The standard errors are given in parentheses.
Table 5: Delta Vega Hedging Errors for Barrier Options

<table>
<thead>
<tr>
<th>Money-ness ((X/F))</th>
<th>Days To Maturity</th>
<th>Absolute Errors</th>
<th>Dollar Errors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>BS</td>
<td>CEV</td>
</tr>
<tr>
<td>0.94</td>
<td>60</td>
<td>0.3393</td>
<td>0.3262</td>
</tr>
<tr>
<td>0.96</td>
<td></td>
<td>(0.638)</td>
<td>(0.648)</td>
</tr>
<tr>
<td>0.98</td>
<td></td>
<td>0.2681</td>
<td>0.2798</td>
</tr>
<tr>
<td>1.00</td>
<td></td>
<td>(0.500)</td>
<td>(0.514)</td>
</tr>
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<td>1.02</td>
<td></td>
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<td>0.2686</td>
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<td>Dollar Errors</td>
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<td>BS CEV JUMP SV</td>
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<td>(0.215) (0.474) (1.454) (1.203)</td>
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<td>-0.1976 -0.2050 -0.0926 -0.1590</td>
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<td></td>
<td>(0.313) (0.668) (2.314) (2.119)</td>
<td>(0.322) (0.805) (3.293) (2.842)</td>
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<tr>
<td>0.96</td>
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<td>0.1851 0.4252 2.2390 1.7710</td>
<td>-0.1707 -0.1760 -0.0640 -0.1887</td>
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<td>(0.277) (0.573) (2.204) (2.007)</td>
<td>(0.286) (0.691) (3.140) (2.670)</td>
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<td>(0.233) (0.473) (2.036) (1.880)</td>
<td>(0.243) (0.575) (2.870) (2.670)</td>
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<td>(0.184) (0.371) (1.768) (1.603)</td>
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<td>(0.134) (0.272) (1.424) (1.300)</td>
<td>(0.150) (0.343) (1.862) (1.634)</td>
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<td>(0.088) (0.186) (1.044) (0.994)</td>
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<tr>
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<td>(0.041) (0.106) (0.570) (0.793)</td>
<td>(0.043) (0.132) (0.657) (0.872)</td>
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</table>

Note: Table 5 reports the delta-vega hedging errors for barrier call options. Days-to-maturity are actual days. The hedge portfolios are rebalanced daily. The standard errors are given in parentheses.
### Table 6: Delta Vega Hedging Errors for Compound Options

<table>
<thead>
<tr>
<th>Strike Price ($X_t$)</th>
<th>Days To Maturity</th>
<th>Absolute Errors</th>
<th>Dollar Errors</th>
</tr>
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<td></td>
<td></td>
<td>BS</td>
<td>CEV</td>
</tr>
<tr>
<td>1.5</td>
<td>60</td>
<td>0.6430</td>
<td>0.6737</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.532) (0.561) (0.578) (0.771)</td>
<td>(0.834) (0.876) (0.845) (0.947)</td>
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<tr>
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<td>0.5834</td>
<td>0.6102</td>
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<tr>
<td></td>
<td></td>
<td>(0.489) (0.515) (0.554) (0.775)</td>
<td>(0.760) (0.797) (0.804) (0.901)</td>
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<tr>
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<td>180</td>
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<td>0.5028</td>
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<tr>
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<td></td>
<td>(0.429) (0.451) (0.451) (0.747)</td>
<td>(0.760) (0.674) (0.618) (0.832)</td>
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<td>0.3874</td>
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<td>(0.366) (0.383) (0.321) (0.701)</td>
<td>(0.520) (0.543) (0.415) (0.766)</td>
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<td>(0.302) (0.316) (0.224) (0.617)</td>
<td>(0.405) (0.423) (0.280) (0.669)</td>
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<td>(0.240) (0.252) (0.150) (0.500)</td>
<td>(0.306) (0.322) (0.188) (0.538)</td>
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<td>(0.576) (0.606) (0.649) (1.106)</td>
<td>(0.922) (0.967) (0.985) (1.303)</td>
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<td>(0.537) (0.564) (0.612) (1.061)</td>
<td>(0.858) (0.898) (0.926) (1.207)</td>
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<td>(0.486) (0.519) (0.709) (0.990)</td>
<td>(0.760) (0.794) (0.861) (1.102)</td>
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<td>(0.434) (0.454) (0.424) (0.915)</td>
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<td>(0.332) (0.348) (0.219) (0.683)</td>
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<td>(0.740) (0.770) (0.967) (1.460)</td>
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<td>(0.676) (0.702) (0.827) (1.456)</td>
<td>(1.059) (1.097) (1.245) (1.653)</td>
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<td>(0.637) (0.662) (0.731) (1.459)</td>
<td>(0.983) (1.017) (1.059) (1.663)</td>
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<td>(0.557) (0.579) (0.529) (1.388)</td>
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Table 6-Continued

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Note: Table 6 reports the delta-vega hedging errors for call on call options. Days-to-maturity are actual days. The hedge portfolios are rebalanced daily. The standard errors are given in parentheses.