Estimating Structural Credit Risk Models with Consideration of Survivorship

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Abstract

One critical difficulty in implementing structural credit risk models is that the underlying asset value cannot be directly observed. Models require the unobserved asset value and the unknown parameter(s) as inputs; for example, asset value and volatility are in practice unknown when the model of Merton (1974) is applied. The estimation problem is further complicated by the fact that typical data samples are for the survived firms. This paper applies the maximum likelihood principle to develop an estimation procedure. The maximum likelihood estimator for parameter(s), asset value, credit spread and default probability are derived for Merton’s model. A Monte Carlo study is conducted to examine the performance of the maximum likelihood method. An application to real data is also presented.

Keywords: Credit risk, maximum likelihood, option pricing, Monte Carlo simulation.

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1 Introduction

In Merton (1974), a pricing model for corporate liabilities was developed using an option valuation approach. In his setting, the unobserved asset value of the firm is governed by a geometric Brownian motion. Subsequently, many variants of this model have been proposed in the literature. Merton’s and its extended models are typically referred to as structural credit risk (or risky bond) models. Examples abound; Longstaff and Schwartz (1995), Madan and Unal (2000) and Collin-Dufresne and Goldstein (2001). This paper develops a maximum likelihood estimation method for this class of models, specifically for the deterministic model of Merton (1974).

As pointed out in Jarrow and Turnbull (2000) among others, there are several limitations associated with the implementation of structural credit spread models. First, the asset value is an unobserved quantity. This in turn creates problems with the estimation of the various required parameters such as the drift and volatility of the asset value process and the correlation among different asset value processes.

In the academic literature, two approaches have been proposed for dealing with the estimation problem when the underlying asset value is unobserved. The first approach, which we will refer to as the implicit estimation method, uses some observed quantities and the corresponding restrictions derived from the theoretical model to extract point estimates for the model parameter(s) and the unobserved asset value. Take the univariate case of Merton’s (1974) model as an example. The implicit estimation method relies on two equations: one relating the equity value to the asset value and the other relating the equity volatility to the asset volatility. The two-equation system can then be solved for the two unknown variables: the asset value and volatility. The implicit estimation method has been adopted by Ronn and Verma (1984) to implement the deposit insurance pricing model of Merton (1978) and by Jones, Mason and Rosenfeld (1984) to conduct an empirical study of Merton’s (1974) risky bond pricing model. A three-equation extension of the implicit estimation method was used in Duan, Moreau and Sealey (1995) to implement their deposit insurance model with stochastic interest rate where the third equation relates the equity duration to the asset volatility.

The second estimation approach was proposed by Duan (1994, 2000). A likelihood function based on the observed equity values is derived by employing the transformed data principle in conjunction with the equity pricing equation. With the likelihood function in place, maximum
likelihood estimation and statistical inference become straightforward. The maximum likelihood method was applied to Merton’s (1978) deposit insurance pricing model in Duan (1994), Duan and Yu (1994), and Laeven (2002). Later, Duan and Simonato (2002) extended the method to deposit insurance pricing under stochastic interest rate. For credit risk, the estimation method has been applied to a strategic corporate bond pricing model by Ericsson and Reneby (2001).

This paper is in line with the second estimation approach. Different from the existing works, we explicitly take into account the survivorship issue. In the credit risk setting, it is imperative for analysts to recognize the fact that a firm in operation has by definition survived thus far. Estimating a credit risk model using the sample of equity prices needs to reflect this reality, or runs the risk of biasing the estimator. This contrasts interestingly with the deposit insurance setting for which survived banks may have actually failed but continue to stay afloat due to deposit insurance.

To our knowledge, the maximum likelihood estimation method, except for Ericsson and Reneby (2001), has not been applied to the credit spread models. Our paper is the first to address the survivorship issue as well as applying the method to a portfolio context of credit risk assessment. For credit risk, it is essential to correctly assess correlations because standard credit risk management methods such as CreditMetrics and KMV are highly sensitive to the correlation coefficients of asset returns (see Crouhy and Mark (1998)).

Theoretically, the maximum likelihood estimation method has several advantages compared to the implicit estimation method. First, the maximum likelihood method provides an estimate of the drift of the unobserved asset value process under the physical probability measure. This can in turn be used to obtain an estimate of the default probability of the firm. Such an estimate is not available within the context of the implicit estimation method since the theoretical equity pricing equation does not contain the drift of the asset value process under the physical probability measure. The second advantage is associated with the asymptotic properties of the maximum likelihood estimator such as consistency and asymptotic normality, which in turn allows for statistical inference to assess the quality of parameter estimates and/or perform testing on the hypotheses of interest. In contrast, consistency is unattainable with the implicit estimation method because it erroneously forces a stochastic variable to be a constant (see Duan (1994)).

We develop the likelihood function and implement the maximum likelihood estimation procedure for Merton’s (1974) model. We also perform a Monte Carlo study to ascertain the method’s performance for a reasonable sample size. The bias caused by neglecting survivorship is examined.
Finally, we apply the analysis to real data of two firms.

2 Merton’s credit risk model

In the Merton (1974) framework, firms have a very simple capital structure. It is assumed that the $i$th firm is financed by equity with a market value $S_{i,t}$ at time $t$ and a zero-coupon debt instrument with a face value of $F_i$ maturing at time $T_i$. Let $V_{i,t}$ be the asset value of the firm and $D_{i,t}(\sigma_{V_i})$ its risky zero-coupon bond value at time $t$. Let us consider $m$ firms. Naturally, the following accounting identity holds for every time point and for every firm:

$$V_{i,t} = S_{i,t} + D_{i,t}(\sigma_{V_i}), \text{ for any } t \geq 0 \text{ and } i = 1, \cdots, m.$$ (1)

It is further assumed that the asset values follow geometric Brownian motions; that is, on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, P)$, we have

$$dV_{i,t} = \mu_i V_{i,t} dt + \sigma_{V_i} V_{i,t} dW_{i,t}, \ i \in \{1, \cdots, m\}$$ (2)

where $\mu_i$ and $\sigma_{V_i}$ are, respectively, the drift and diffusion coefficients under the physical probability measure $P$ and the $m$ dimensional Brownian motion $W = \{(W_{1,t}, \cdots, W_{m,t}) : t \geq 0\}$ is such that

$$\text{Cov}^P [W_{i,t}, W_{j,t}] = \rho_{ij} t \text{ for any } t \geq 0.$$ (3)

The default-free interest rate $r$ is assumed to be a constant. The default of the $i$th firm occurs at time $T_i$ if the asset value $V_{i,T_i}$ is below the face value $F_i$ of the debt. Using these assumptions, formulas for the bond value and default probability can be obtained. These formulas are provided in Appendix A. The credit spread formula follows immediately from the formula for $D_{i,t}(\sigma_{V_i})$. Because the default-free interest rate is a constant, the credit spread can be written as

$$C_{i,t}(\sigma_{V_i}) = -\frac{\ln[D_{i,t}(\sigma_{V_i})/F_i]}{T_i - t} - r.$$ (4)

To implement this model empirically, one needs values for the input variables. Specifically, for the $i$th firm, the asset value $V_{i,t}$, its drift $\mu_i$ and its diffusion coefficient $\sigma_{V_i}$ are unknown. The correlation coefficient between any two asset values is also unknown. In the next sub-section, we derive the likelihood function which serves as the basis for maximum likelihood estimation.
2.1 The likelihood function

The specific idea for constructing the likelihood function is taken from Duan (1994, 2000) which treats the observed time series of equity prices as a sample of transformed data with the equity pricing equation defining the transformation. Loosely speaking, the resulting likelihood function becomes the likelihood function of the implied asset values multiplied by the Jacobian of the transformation evaluated at the implied asset values.

In general, one observes a time series of equity values for the $i$th firm corresponding to a known face value of debt over a sample period with a time step of length $h$. Denote the time series sample up to time $t$ by $\{s_{i,0}, s_{i,h}, s_{i,2h}, \cdots, s_{i,Nh}\}$ with $t = Nh$ and $i \in \{1, \cdots, m\}$. The observation period is assumed to be the same for all firms to facilitate the estimation of the correlation coefficients. Let $\theta$ denote the vector containing the parameters associated with the $m$-dimensional geometric Brownian motion process; that is,

$$\theta = [\mu_1, \cdots, \mu_m, \sigma_{V_1}, \cdots, \sigma_{V_m}, \rho_{12}, \cdots, \rho_{1m}, \rho_{23}, \cdots, \rho_{2m}, \cdots]. \quad (5)$$

The function defining the critical transformation is the equity pricing equation which is $S_{i,t} = V_{i,t} - D_{i,t}(\sigma_{V_i})$. It can be easily shown that $S_{i,t}$ is an invertible function of $V_{i,t}$ for any $\sigma_{V_i}$. We denote it by $S_{i,t} = g_i(V_{i,t}; t, T, \sigma_{V_i})$.

The data sample may contain firms that have refinanced during the sample period. Assessing credit spreads is an exercise of attaching a premium to debt instruments of a firm that has survived. The fact that a firm survives needs to be incorporated into the likelihood function. Assume there are refinancing points in the sample, denoted by $n_1h < n_2h < \cdots < n_ch \leq Nh$. Refinancing occurs in the sample period whenever the zero-coupon debt of at least one firm reaches its maturity and new zero-coupon debts are issued. Survivorship is not the relevant issue in the case of Merton’s (1974) model if the zero-coupon debt did not come due during the sample period because there would be no possibility of defaulting on the debt obligations. If the data sample contains points of refinancing, survivorship adjustment is important. For purposes of addressing survivorship, we let $D$ be the event “no default for the entire sample period”. The relevant log-likelihood function is given in the following theorem.

**Theorem 1** Assume that refinancing took place but there was no default in the sample period. The
log-likelihood function corresponding to the stock price sample is

\[
L(s_0, s_h, s_{2h}, \ldots, s_{Nh}; \theta) = -\frac{mN}{2} \ln (2\pi) - \frac{N}{2} \ln (\det \Sigma) - \frac{1}{2} \sum_{k=1}^{N} w_{kh}^* \Sigma^{-1} w_{kh}^* - \sum_{k=1}^{N} \sum_{i=1}^{m} \ln v_{i,kh}^*
\]

where \( v_{i,kh}^* = g_i^{-1}(s_{i,kh}; \sigma_{V_i}, F_{i,kh}, T_{i,kh}) \) is the asset value implied by the equity value, \( F_{i,kt} \) and \( T_{i,kt} \) are the face value and maturity date of the debt for the \( i \)th firm at time \( t \), \( \Phi(\cdot) \) is the standard normal distribution function, \( d(\cdot, \cdot, \cdot) \) is defined in equation (16) of Appendix A, \( w_{kh}^* \) is an \( m \)-dimensional column vector defined as

\[
w_{kh}^* = \left( \ln v_{i,kh}^* - \ln v_{i,(k-1)h}^* - \left( \mu_i - \frac{1}{2} \sigma_{V_i}^2 \right) h \right)_{m \times 1}
\]

and

\[
\Sigma = \begin{bmatrix}
\sigma_{V_1}^2 & \cdots & \sigma_{V_1} \sigma_{V_m} \rho_{1m} \\
\vdots & \ddots & \vdots \\
\sigma_{V_1} \sigma_{V_m} \rho_{1m} & \cdots & \sigma_{V_m}^2
\end{bmatrix} h
\]

with \( \det \Sigma \) being the determinant of \( \Sigma \). Moreover, \( V_j = \bigcap_{i=1}^{m} \{ v_{i,n_jh} > F_{i,n_jh}1_{T_{i,n_jh}=n_jh} \} \subseteq \mathbb{R}^m \) is the subset of the sample space at time \( n_jh \) corresponding to no default and \( P(D; \theta) \) is the survival probability according to equation (19).

The proof for this theorem is given in Appendix B. The first four terms on the right hand side of equation (2) constitute the log-likelihood function if the asset values were observed and no refinancing took place in the sample period. The fifth term in equation (6) corresponds to the Jacobian that accounts for the transformation from the observed equity values to the implied asset values. It is important to note that \( v_{i,kh}^* \) depends on the parameters of the model. If it were not, \( \sum_{k=1}^{N} \sum_{i=1}^{m} \ln v_{i,kh}^* \) could be dropped from the likelihood function. This would be the case if the asset value could be directly observed. Finally, the survival probability \( P(D; \theta) \) reflects the fact that the log-likelihood function is conditional on no default in the sample period whereas the term \( \sum_{j=1}^{c} \ln 1_{v_{n_jh}^* \in V_j} \) simply assigns a zero likelihood (or log-likelihood equal to minus infinity) to the parameter value at which some implied asset value suggesting a default.

If the firms in the sample never faced refinancing during the sample period, survivorship is not an issue. In terms of the above theorem, both \( \ln P(D; \theta) = 0 \) and \( \sum_{j=1}^{c} \ln 1_{v_{n_jh}^* \in V_j} = 0 \) because
the survival probability equals 1 and $1_{V_{j,k} \in V_j} = 1$ when there is no refinancing. This reduced case actually amounts to a straightforward generalization of Duan (1994, 2000) to a portfolio context. With the likelihood function in place, one can conduct the maximum likelihood estimation and statistical inference.

2.2 The estimation procedure

Although directly maximizing the log-likelihood function seems a natural approach, it is actually not practical when many firms are in the data sample. The number of parameters involved increases rapidly and quickly becomes unmanageable. We therefore adopt the following multi-step estimation procedure, knowing that the true optimum may not be obtained this way:

- **Step 1:** Estimate the Merton (1974) model for each firm separately. For firm $i$, estimates $\hat{\mu}_i$ and $\hat{\sigma}_{V_i}$ are obtained using the log-likelihood function in (2) by imposing $m = 1$. The Monte Carlo study (see Section 2.3) indicates that the following standard asymptotic results are applicable to the typical application sample size.

  **Statement:** The parameter estimates $(\hat{\mu}_i, \hat{\sigma}_{V_i})$ are asymptotically normally distributed around the true parameter values with the covariance matrix being approximated by $\hat{F}_i^{-1}$ where

  $$\hat{F}_i = \left( \begin{array}{cc} -\frac{1}{N} & \frac{1}{N} \\
 0 & -\frac{1}{N} \end{array} \right) \left( \begin{array}{cc} \frac{\partial^2 L}{\partial \mu_i^2} & \frac{\partial^2 L}{\partial \mu_i \partial \sigma_{V_i}} \\
 \frac{\partial^2 L}{\partial \mu_i \partial \sigma_{V_i}} & \frac{\partial^2 L}{\partial \sigma_{V_i}^2} \end{array} \right).
  $$

- **Step 2:** Compute the implied firm value by

  $$\hat{V}_{i,t} \equiv V_{i,t}(\hat{\sigma}_{V_i}) = g_i^{-1}(S_{i,t}; t, \hat{\sigma}_{V_i}).$$

  Because $V_{i,t}(\hat{\sigma}_{V_i})$ is a continuously differentiable function of $\hat{\sigma}_{V_i}$, the distribution for the firm’s asset value can be approximated by a normal distribution (see Lo (1986) or Rao (1973), page 385). Let $\hat{\nabla}_{V_i} = \left( \frac{\partial V_{i,t}(\hat{\sigma}_{V_i})}{\partial \mu_i}, \frac{\partial V_{i,t}(\hat{\sigma}_{V_i})}{\partial \sigma_{V_i}} \right)$, then

  $$V_{i,t}(\hat{\sigma}_{V_i}) - \hat{V}_{i,t} \sim N \left\{ 0, \hat{\nabla}_{V_i} \hat{F}_i^{-1} \hat{\nabla}_{V_i} \right\}.$$  

  \(^1\)Note that the equity pricing formula does not depend on $\mu_i$. Thus, $\frac{\partial V_{i,t}(\hat{\sigma}_{V_i})}{\partial \mu_i} = 0$. Moreover,

  $$\frac{\partial V_{i,t}(\hat{\sigma}_{V_i})}{\partial \sigma_{V_i}} = \frac{1}{V_{i,t}(t) \sqrt{T-t} \phi \left( d \left( \hat{V}_{i,t}, t, \hat{\sigma}_{V_i} \right) \right)}$$

  where $\phi(\cdot)$ denotes the standard normal density function.
For the credit spread, recall equation (4). Its point estimate can be computed by

$$C_{i,t}(\hat{\sigma}_{V_i}) = -\frac{\ln \left( \frac{V_{i,t}(\hat{\sigma}_{V_i}) - S_{i,t}}{F_t} \right)}{T_i - t} - r$$

and its distribution can be approximated by

$$C_{i,t}(\hat{\sigma}_{V_i}) - C_{i,t}(\sigma_{V_i}) \sim N \left\{ \nabla C_{i,t} \hat{F}_t^{-1} \nabla' \right\}$$

where $\hat{\nabla} C_{i,t} = \left( \frac{\partial C_{i,t}(\hat{\sigma}_{V_i})}{\partial \mu_i}, \frac{\partial C_{i,t}(\hat{\sigma}_{V_i})}{\partial \sigma_{V_i}} \right)$. The default probability is a function of both $\mu_i$ and $\sigma_{V_i}$ and its expression is given in Appendix A. The point estimate can thus be expressed as

$$P_{i,t}(\hat{\mu}_i, \hat{\sigma}_{V_i}) = \Phi \left( x_{i,t}(\hat{\mu}_i, \hat{\sigma}_{V_i}, V_{i,t}) \right)$$

where $\Phi$ is the standard normal distribution function and

$$x_{i,t}(\hat{\mu}_i, \hat{\sigma}_{V_i}) = \frac{\ln (F_t) - \ln (V_{i,t}(\hat{\sigma}_{V_i})) - \left( \hat{\mu}_i - \frac{1}{2} \hat{\sigma}_{V_i}^2 \right) (T_i - t)}{\hat{\sigma}_{V_i} \sqrt{T_i - t}}.$$

The distribution for the default probability estimate should be treated with care. A direct application of the first-order Taylor approximation as the typical asymptotic theory calls for is not an advisable approach to this particular estimator. Because the sampling error associated with $\hat{\mu}_i$ is quite large and the mapping from $(\hat{\mu}_i, \hat{\sigma}_{V_i})$ to $P_{i,t}(\hat{\mu}_i, \hat{\sigma}_{V_i})$ is highly non-linear, we thus have adopted a two-step construction and found it work well. First, we use the first-order Taylor approximation for $x_{i,t}(\hat{\mu}_i, \hat{\sigma}_{V_i})$; that is,

$$x_{i,t}(\hat{\mu}_i, \hat{\sigma}_{V_i}) - x_{i,t}(\mu_i, \sigma_{V_i}) \sim N \left\{ \nabla x_{i,t} \hat{F}_t^{-1} \nabla' \right\}$$

where $\hat{\nabla} x_{i,t} = \left( \frac{\partial x_{i,t}(\hat{\mu}_i, \hat{\sigma}_{V_i})}{\partial \mu_i}, \frac{\partial x_{i,t}(\hat{\mu}_i, \hat{\sigma}_{V_i})}{\partial \sigma_{V_i}} \right)$. The $1 - \alpha$ confidence interval for the default probability $P_{i,t}(\hat{\mu}_i, \hat{\sigma}_{V_i}) = \Phi \left( x_{i,t}(\hat{\mu}_i, \hat{\sigma}_{V_i}) \right)$ is then constructed as:

$$\left[ \Phi \left( b \left( \hat{\mu}_i, \hat{\sigma}_{V_i}, V_{i,t} \right) \right); \Phi \left( c \left( \hat{\mu}_i, \hat{\sigma}_{V_i}, V_{i,t} \right) \right) \right]$$

\(^2\)Again, the first component of $\hat{\nabla} C_{i,t}$ is zero. The second one is

$$\frac{\partial C_{i,t}(\hat{\sigma}_{V_i})}{\partial \sigma_{V_i}} = -\frac{1}{T_i - t} \frac{1}{V_{i,t}(\hat{\sigma}_{V_i}) - S_{i,t}} \frac{\partial V_{i,t}(\hat{\sigma}_{V_i})}{\partial \sigma_{V_i}},$$

where $\frac{\partial V_{i,t}(\hat{\sigma}_{V_i})}{\partial \sigma_{V_i}}$ is given in an earlier footnote.
where $\underline{b}(\bullet)$ and $\bar{b}(\bullet)$ are the lower and upper bounds of the $1 - \alpha$ confidence interval for $x_{i,t}(\hat{\mu}_i, \hat{\sigma}_{V_i})$. The validity of this confidence interval can be easily verified as follows:

$$1 - \alpha = P[\underline{b}(\hat{\mu}_i, \hat{\sigma}_{V_i}) \leq x_{i,t}(\hat{\mu}_i, \hat{\sigma}_{V_i}) \leq \bar{b}(\hat{\mu}_i, \hat{\sigma}_{V_i})] = P[\Phi(\underline{b}(\hat{\mu}_i, \hat{\sigma}_{V_i})) \leq \Phi(x_{i,t}(\hat{\mu}_i, \hat{\sigma}_{V_i})) \leq \Phi(\bar{b}(\hat{\mu}_i, \hat{\sigma}_{V_i}))].$$

**Step 3:** Compute the sample correlation coefficient between $\ln(\hat{V}_{i,kh}/\hat{V}_{i,(k-1)h})$ and $\ln(\hat{V}_{j,kh}/\hat{V}_{j,(k-1)h})$ and use it as the estimate for $\rho_{ij}$. The estimated correlation coefficient is expected to distribute normally around its true value with a variance taken from the corresponding diagonal entry of $\hat{F}_{ij}^{-1}$ where

$$\hat{F}_{ij} = -\frac{1}{N} \frac{\partial^2 L(\bullet)}{\partial \theta_{ij}(k) \partial \theta_{ij}(l)} \Big|_{\theta_{ij} = \hat{\theta}_{ij}} = [\mu_i, \mu_j, \sigma_{V_i}, \sigma_{V_j}, \rho_{ij}],$$

$\theta_{ij}(k)$ is the $k$th element of $\theta_{ij}$, $\hat{\theta}_{ij}$ is the estimate for $\theta_{ij}$ and $L(\cdot)$ is the joint log-likelihood function of the data sample for the $i$th and $j$th firms. Note that the first four entries of $\hat{\theta}_{ij}$ are taken from the individual estimations for the $i$th and $j$th firms in Step 1 whereas the correlation coefficient estimate is obtained in this step. For any quantity that is a function of $\theta_{ij}$, the variance of its distribution can be obtained using the whole matrix $\hat{F}_{ij}^{-1}$ in a way similar to those in Step 2. If one is interested in any quantity that is a function of the parameters for more than two firms, the dimension of $\hat{F}_{ij}$ can expanded to accommodate the new requirement.

The three-step estimation procedure can be completed fairly quickly; for example, on a standard desktop computer, the completion for two firms usually takes approximately 10 seconds. Parameter estimation for a large portfolio of firms is thus feasible with the three-step estimation procedure. The numerical optimization routine used here is the quadratic hill–climbing algorithm of Goldfeld, Quandt and Trotter (1966) with a convergence criterion based on the absolute values of the changes in parameter values and functional values between successive iterations. We consider convergence achieved when both of these changes are smaller than $10^{-5}$. The three-step estimation procedure uses several simplifications. We need to ascertain its performance by a Monte Carlo study. Such a study is carried out in the next sub-section.
2.3 A Monte Carlo study

In order to assess the quality of the maximum likelihood procedure, we examine how well the normal distribution suggested by the theory approximates the actual distribution for a reasonable sample size. In other words, we verify whether the parameter estimates for a sample size \( N \) is well approximated by the distribution given in the preceding subsection. Similarly, we check the distributions for the asset value \( V_{i,t}(\hat{\theta}_N) \), credit spread \( C_{i,t}(\hat{\theta}_N) \) and default probability \( P_{i,t}(\hat{\theta}_N) \).

We consider the case of two firms and simulate the data on a daily basis as follows:

- **Step 1:** Let \( V_{i,kh} \) for \( i = \{1, 2\} \) and \( k = \{1, \ldots, N\} \) denote the simulated asset values in accordance with
  \[
  V_{1,(k+1)h} = V_{1,kh} \exp \left( \mu_1 h - \frac{1}{2} \sigma_{V_1}^2 h + \sigma_{V_1} \sqrt{h} \epsilon_{1,k} \right)
  \]
  \[
  V_{2,(k+1)h} = V_{2,kh} \exp \left( \mu_2 h - \frac{1}{2} \sigma_{V_2}^2 h + \sigma_{V_2} \sqrt{h} \epsilon_{2,k} \right)
  \]
  where \( \{ (\epsilon_{1,k}, \epsilon_{2,k})' : k \in \{1, \cdots, N\} \} \) is a sequence of independent and identically distributed vectors of standard normal random variables with a correlation coefficient of \( \rho_{12} \). To be consistent with daily data, we set \( h = 1/250 \). The specific parameter values used in the simulation are given in the table.

- **Step 2:** Use the simulated asset values to compute equity values by the equity pricing equation in Appendix B. If the zero-coupon debt of any one of these two firms comes due at any time point, its equity value may become zero. If that occurs, the firm is deemed at default and that particular sample is discarded. In other words, we only keep a simulated samples of equity prices if both firms survive the entire period.

For each simulated data sample, we conduct maximum likelihood estimation and compute the point estimates and their associated variances based on the equity prices. We repeat the simulation run 5,000 times to obtain the Monte Carlo estimates for the relevant quantities. We consider two simulation scenarios.

- **Scenario 1:** Both firms have a debt maturity beyond the sample period. In other words, there will be no refinancing in the sample period and hence no need to consider the survivorship issue. Specifically, we have \( T = 3, t = 2, h = 1/252 \) and \( N = 504 \). This means that we
simulate data daily for two years and at the beginning of the simulated sample, both firms have zero-coupon debts with three-year maturity.

- **Scenario 2:** Both firms will experience refinancing but at different time points. We again set $t = 2, h = 1/252$ and $N = 504$. Both firms refinance with one-year zero coupon debts. Firm 1 has a zero-coupon debt with 0.25 years to maturity at the beginning of the data period whereas firm 2 has 0.75 years to maturity at that point. Thus, firm 1 has two refinancing points at 0.25 and 1.25 years. For firm 2, the two refinancing points are 0.75 and 1.75 years. One needs to be particularly careful in dealing with refinancing. At a refinancing point, the market value of the new zero-coupon debt is lower than its face value. If we maintain the same face value, the amount of the newly raised debt financing will be less than the payment required to retire the old debt. As a result, the firm’s asset value must be adjusted downward to reflect the extra funds needed. Alternatively, one can increase the face value of the new debt so that the new debt financing leaves the firm’s asset value unchanged. Still possible, one can set the face value of the new debt to a level that the new debt-to-asset-value ratio remains the same as the one at the previous refinancing point. All these possibilities requires of factoring in the valuation effect on the new zero-coupon debt, and effectively imposes the self-financing constraint. It is the last approach that we have adopted for the simulation study, which means the same debt-to-asset-value ratio at all refinancing time points for the survived firm.

Four different cases are examined. Each case is for a combination of high or low debt-to-asset-value ratio ($F_i/V_{i,0}$) with high or low noise-to-signal ratio ($\sigma_{V_i}/\mu_i$). For Scenario 1 (guaranteed survival), we only report the high-high case in Table 1 to conserve space. In all these experiments, we set $r = 0.05$ and $\rho_{ij} = 0.5$. For a given simulation, the values of $V_{i,t}$ and $P_{i,t}(\mu_i, \sigma_{V_i})$ depend on the specific sample path taken by $V_{i,t}$, we thus examine the statistics for the difference between the estimate and the true value.

Table 1 reports the simulation results for the high-high case with $F_i/V_{i,0} = 0.9$ and $\sigma_{V_i}/\mu_i = 3$ for both firms when survival is guaranteed. The results reveal that for all parameters and the inferred variables, the maximum likelihood estimators are unbiased (except in the case of default probability). For the default probability, there appears to be an upward bias but the median
is correctly located, indicating the presence of skewness. The coverage rates\(^3\) indicate that the asymptotic distribution approximates well well the small sample distribution. In this table, we have also reported the results using the implicit estimation method. The description of this method is given in Appendix C. It is clear from this table that the implicit estimation produces poor estimates. The estimates for volatility are significantly biased downward. For asset value, they are biased upward substantially. For the inference on biases, the standard deviations reported in the tables should be divided by a factor of \(\sqrt{5000}\) to reflect the fact that 5000 simulation runs are used to produce the means. The performance of the implicit estimation method is related to the noise-to-signal ratio of the firm. For high noise-to-signal firms, the biases are very pronounced whereas for low noise-to-signal firms, the estimates (not reported here) may be regarded as acceptable. Also worth noting is that the implicit estimation method cannot yield an estimate for the drift coefficient and consequently it cannot provide an estimate for the default probability. On the efficiency side, the maximum likelihood estimators always show a smaller standard deviation when compared to the implicit estimators. For all parameter estimates and inferred variables, except for the correlation coefficient, the standard deviations of the implicit estimators are approximately 10 times larger than their maximum likelihood counterparts. The correlation estimate based on the implicit estimation method is very good in terms of bias and standard deviation. This indicates that, in the context of Merton’s (1974) model, the correlation between the stock price return is a very good estimate of the true correlation between the asset returns. This is perhaps not too surprising because the quadratic variation process between \(\ln S_i\) and \(\ln S_j\) is a function of the correlation between the two Wiener processes driving the asset value processes. Normalizing this expression by the sample standard deviation of the stock returns thus gives an estimate that is close to the true correlation.

[Scenario 2 simulation to be completed later]

2.4 Empirical analysis

In this section, we implement Merton’s (1974) model using real data. Although this model assumes a zero-coupon debt, most corporations have much more complex liability structures. Liabilities with different properties such as maturity, seniority and coupon rate must be aggregated into one quantity to implement the model. Obviously, there is no clear-cut solution to these problems. One

\(^3\)A coverage rate is the percentage of the parameter estimates for which the true parameter value is contained in the \(\alpha\) confidence interval implied by the asymptotic distribution
possible approach to determining debt maturity is to find a “theoretical” zero-coupon bond that has the same duration as the aggregated debt. But doing so fundamentally changes the pattern of cash flows. Another way of addressing the issue is to argue that the annual report on profits and losses is perceived by equity holders as the maturity date of their option, which then leads to a pseudo debt maturity of one year. At the time of the public reporting of the annual profits and losses, debt holders may decide to take control of the firm in case of insolvency. On the other hand, if the firm is solvent, the equity value equals the difference between the asset value and the face value of debts.

Determining the amount of debt for the model is also not an obvious matter. The simplest approach would be to set the face value of debt equal to the total amount of short- and long-term liabilities. However, as argued in Crouhy, et al. (2000) where some details regarding the implementation of the KMV method are reported, the probability of the asset value falling below the total face value of liabilities may not be an accurate measure of the actual default probability. As reported in an empirical study by KMV, firms default when the asset value reaches a level somewhere between the face value of total liabilities and the face value of the short-term debt. Moreover, there are unknown undrawn commitments (lines of credit) which can be used in case of financial distress. These considerations lead to an ad-hoc approach adopted by KMV to set the face value of debt equal to $1/2$ of the long-term debt plus the full amount of the short-term debt.

In this paper, we use the major refinancing time points to determine the maturity and consider survivorship issue. Two companies from the Canadian retailing sector are examined for years 1999-2001: Hudson’s Bay Company and Sears Canada Inc. Their stock prices are taken from DataStream and the information regarding the long- and short-term debts is extracted from the Financial Post Historical Reports. The short-term debt is defined as the “Current Liabilities” reported yearly in the consolidated balance sheet. The long-term debt is the account “Long-term Debt, net” which represents the long-term debt net of long-term debt maturing during the current year.

[To be completed later.]

3 Conclusion

[To be completed later.]
A Formulas for debt and default probability in Merton’s (1974) model

In Merton (1974), \( D_{i,T_i}(\sigma_{V_i}) = \min \{ V_{i,T_i}, F_i \} \). For valuation, it is well-known that we can use the risk-neutral asset price dynamic to evaluate the discounted expected payout, where the risk-neutral dynamic has the risk-free rate as the drift term but the same diffusion term. Consequently, the bond value at time \( t \) is

\[
D_{i,t}(\sigma_{V_i}) = F_i e^{-r(T_i-t)} \left( \frac{V_{i,t}}{F_i e^{-r(T_i-t)}} \Phi(-d(V_{i,t}, t, \sigma_{V_i})) + \Phi \left( d(V_{i,t}, t, \sigma_{V_i}) - \sigma_{V_i} \sqrt{T_i-t} \right) \right) \tag{15}
\]

where \( \Phi(\bullet) \) is the standard normal distribution function and

\[
d(V_{i,t}, t, \sigma_{V_i}) = \frac{\ln(V_{i,t}) - \ln(F_i) + (r + \frac{1}{2} \sigma_{V_i}^2)(T_i-t)}{\sigma_{V_i} \sqrt{T_i-t}}. \tag{16}
\]

Furthermore, Merton’s model implies the following default probability under measure \( P \) at time \( t \):

\[
P_{i,t}(\mu_i, \sigma_{V_i}) = P[V_i(T_i) < F_i | \mathcal{F}_t] = \Phi \left( \frac{\ln(F_i) - \ln(V_{i,t}) - (\mu_i - \frac{1}{2} \sigma_{V_i}^2)(T_i-t)}{\sigma_{V_i} \sqrt{T_i-t}} \right). \tag{17}
\]

where \( P[\bullet | \mathcal{F}_t] \) denotes for the conditional probability taken at time \( t \) under the measure \( P \). The joint probability of default for several firms can be expressed using the multivariate normal cumulative distribution function \( N_{0,\rho} : \mathbb{R}^m \rightarrow [0,1] \) with mean \( 0_{m \times 1} \) and covariance matrix \( \rho \equiv (\rho_{ij})_{i,j \in \{1,\ldots,m\}} \) and relying on the following quantity:

\[
P[V_1_{T_1} < \alpha_1, \ldots, V_m_{T_m} < \alpha_m | \mathcal{F}_1] = N_{0,\rho}(\beta_1, \ldots, \beta_m) \tag{18}
\]

where

\[
\beta_i = \frac{\ln(\alpha_i) - \ln(V_{i,t}) - (\mu_i - \frac{1}{2} \sigma_{V_i}^2)(T_i-t)}{\sigma_{V_i} \sqrt{T_i-t}}.
\]

This gives rise to the joint default probability of the firms \( i_1, \ldots, i_k \) where \( k \leq m \) by setting \( \alpha_i = F_i \) for any \( i \in \{i_1, \ldots, i_k\} \) and \( \alpha_i \rightarrow \infty \) for all \( i \notin \{i_1, \ldots, i_k\} \) in equation (18).
B The likelihood function for Merton’s (1974) model with refinancing during the sample period

In this section we derive the likelihood function for the multivariate version of Merton’s (1974) model with refinancing during the sampling period. More precisely, at times \( n_1 h < n_2 h < \ldots < n_c h \leq Nh \), the zero-coupon debt of at least one firm reaches its maturity and is refinanced with the new zero-coupon debt. Let \( D_j \) be the event “no default at time \( n_j h \)” and \( D \) is the event “no default for the whole sample period”. That is,

\[
D_j = \cap_{i=1}^{m} \{ V_{i,n_j h} > F_{i,t} 1_{T_{i,t} = n_j h} \}, \quad j \in \{1, 2, \ldots, c\},
\]

\[
D = \cap_{j=1}^{c} D_j
\]

Recall that \( F_{i,t} \) and \( T_{i,t} \) are the face value and maturity date of the debt for the \( i \)th firm at time \( t \). \( 1_A \) is the indicator function that is equal one if the event \( A \) is realized and zero otherwise. If the firm \( i \), for example, have a zero coupon debt with a face value of \( F_i \) and a maturity date of \( t_j h \) and refinances at time \( t_j h \) with a zero coupon debt with a face value of \( F^* \) and a maturity date of \( T_i > Nh \), then \( F_{i,t} = F_{i,1} 1_{t \leq t_j h} + F^* 1_{t > t_j h} \) and \( T_{i,t} = t_j h 1_{t \leq t_j h} + T_i 1_{t > t_j h} \).

The conditional density function is

\[
f_{V_0, V_h, V_2 h, \ldots, V_N h | D} (v_0, v_h, v_2 h, \ldots, v_N h; \theta) = \frac{f_{V_0, V_h, V_2 h, \ldots, V_N h, D} (v_0, v_h, v_2 h, \ldots, v_N h; \theta)}{P (D; \theta)}
\]

provided that \( P (D; \theta) > 0 \). By the Markov property of the geometrical Brownian motion, the survival probability can be expressed in terms of the multivariate normal cumulative distribution function \( N_{0, \rho} : \mathbb{R}^m \to [0, 1] \) with mean \( 0_{m \times 1} \) and covariance matrix \( \rho \equiv (\rho_{ij})_{i,j \in \{1, \ldots, m\}} \):

\[
P (D; \theta) = \prod_{j=1}^{c} \left[ P \left( V_{1,n_j h} > F_{1,t} 1_{T_{1,n_j h} = n_j h}, \ldots, V_{m,n_j h} > F_{m,t} 1_{T_{m,n_j h} = n_j h} \mid F_{n_j-1 h} \right) \right]
\]

\[
= \prod_{j=1}^{c} N_{0, \rho} (\beta^*_{1,j}, \ldots, \beta^*_{m,j})
\]

(19)

where

\[
\beta^*_{i,j} = - \left( \frac{\ln \left( F_{i,n_j h} \right) - \ln \left( v_{i,n_j-1 h} \right) - \left( \mu_i - \frac{1}{2} \sigma_i^2 \right) (n_j - n_{j-1}) h}{\sigma_i \sqrt{(n_j - n_{j-1}) h}} 1_{T_{i,n_j h} = n_j h} - \infty 1_{T_{i,n_j h} \neq n_j h} \right).
\]
Let \( V_j = \bigcap_{i=1}^{m} \{ v_{i,n_j h} > F_{i,n_j h} 1_{T_{i,n_j h} = n_j h} \} \subseteq \mathbb{R}^m \) be the subset of the sample space at time \( n_j h \) corresponding to no default. The joint density function of the firm values and the survival event is

\[
\hat{f}_{V_0, V_h, V_2h, \ldots, V_{N_h}, D} (v_{0}, v_{h}, v_{2h}, \ldots, v_{NH}, \theta)
\]

\[
= \prod_{j=1}^{c} \left( \prod_{k=n_{j-1}+1}^{n_j} f_{V_{kh} \mid V_{(k-1)h}} (v_{kh} \mid v_{(k-1)h} ; \theta) 1_{v_{n_j h} \in V_j} \right) \left( \prod_{k=n_{c}+1}^{N} f_{V_{kh} \mid V_{(k-1)h}} (v_{kh} \mid v_{(k-1)h} ; \theta) \right)
\]

where \( n_0 = 0 \) and \( f_{V_{kh} \mid V_{(k-1)h}} (v_{kh} \mid v_{(k-1)h} ; \theta) \) denotes the conditional density function of \( V_{kh} \) given \( V_{(k-1)h} \). Note that, the value of \( \prod_{k=N_{c}+1}^{N} f_{V_{kh} \mid V_{(k-1)h}} \) is set to 1 if such a case occurs.

Since the conditional distribution of \( V_{kh} \) given \( V_{(k-1)h} \) is lognormal, we have the following conditional log-likelihood function:

\[
L (v_{0}, v_{h}, v_{2h}, \ldots, v_{NH}; \theta)
\]

\[
= \ln \hat{f}_{V_0, V_h, V_2h, \ldots, V_{N_h}, D} (v_{0}, v_{h}, v_{2h}, \ldots, v_{NH}, \theta)
\]

\[
= \sum_{k=1}^{N} (v_{kh} \mid v_{(k-1)h} ; \theta) + \sum_{j=1}^{c} \ln 1_{v_{n_j h} \in V_j} - \ln P (D; \theta)
\]

\[
= -\frac{mN}{2} \ln (2\pi) - \frac{N}{2} \ln (|\det \Sigma|) - \frac{1}{2} \sum_{k=1}^{N} w_{kh}' \Sigma^{-1} w_{kh} - \sum_{k=1}^{N} \sum_{l=1}^{m} \ln v_{k,lh}
\]

\[
+ \sum_{j=1}^{c} \ln 1_{v_{n_j h} \in V_j} - \ln P (D; \theta)
\]

(20)

where \( \Sigma \equiv \left( h_{i,i,j} \sigma_{ij} \right)_{i,j \in \{1, \ldots, m\}} \) and \( w_{kh} \) is the column vector

\[
w_{kh} \equiv \left( \ln v_{i,kh} - \ln v_{i,(k-1)h} - \left( \mu_i - \frac{1}{2} \sigma_{V_i}^2 \right) h \right)_{m \times 1}.
\]

We of course do not observe \( v_0, v_h, v_{2h}, \ldots, v_{NH} \). Instead, we have a time series of the equity values \( s_0, s_h, s_{2h}, \ldots, s_{NH} \). The equity pricing equation is

\[
S_{i,t} = V_{i,t} \Phi \left( d (V_{i,t}, t, \sigma_{V_i}; F_{i,t}, T_{i,t}) \right) - F_{i} e^{-r(T_{i,t} - t)} \Phi \left( d (V_{i,t}, t, \sigma_{V_i}; F_{i,t}, T_{i,t}) - \sigma_{V_i} \sqrt{T_{i,t} - t} \right)
\]

(21)

where

\[
d (V_{i,t}, t, \sigma_{V_i}; F_{i,t}, T_{i,t}) = \frac{\ln (V_{i,t}) - \ln (F_{i,t}) + \left( r + \frac{1}{2} \sigma_{V_i}^2 \right) (T_{i,t} - t)}{\sigma_{V_i} \sqrt{T_{i,t} - t}}
\]

(22)
Therefore the sub-matrix $J$. Thus, the log-likelihood function based on the sample of observed equity values is $J$ where the Jacobian $f$ \begin{equation}
abla \ln f \left( s_0, s_h, s_{2h}, \ldots, s_{Nh}; \theta \right) = \ln f \left( s_0, s_h, s_{2h}, \ldots, s_{Nh}; \theta \right) + \ln |\det J| \end{equation}
where the Jacobian $J$ of the transformation is a block diagonal matrix $J = (J_{nh})_{n \in \{1, \ldots, N\}}$ and the sub-matrix $J_{kh}$ is the $m \times m$ matrix
\begin{equation}
J_{kh} = \left( \frac{\partial v_{i,kh}^*}{\partial s_{j,kh}} \right)_{i,j \in \{1, \ldots, m\}}.
\end{equation}
One can show
\begin{equation}
\frac{\partial v_{i,kh}^*}{\partial s_{j,kh}} = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{otherwise}
\end{cases}
\end{equation}
Therefore
\begin{equation}
\ln |\det J| = \ln \prod_{k=1}^{N} \prod_{i=1}^{m} \Phi \left( \frac{1}{v_{i,kh}^*} \right) = -N \sum_{k=1}^{N} \sum_{i=1}^{m} \ln \Phi \left( \frac{1}{v_{i,kh}^*} \right).
\end{equation}
Thus, the log-likelihood function based on the sample of observed equity values is
\begin{equation}
L \left( s_0, s_h, s_{2h}, \ldots, s_{Nh}; \theta \right) = -\frac{mN}{2} \ln (2\pi) - \frac{N}{2} \ln (|\Sigma|) - \frac{1}{2} \sum_{k=1}^{N} w_{kk}^* \Sigma^{-1} w_{kk}^* - \frac{N}{2} \sum_{k=1}^{N} \sum_{i=1}^{m} \ln v_{i,kh}^*
+ \sum_{j=1}^{c} \ln \mathbf{1}_{v_{n,jh}^*} \in V_j - \ln P \left( D; \theta \right)
- \sum_{k=1}^{N} \sum_{i=1}^{m} \ln \Phi \left( \frac{1}{v_{i,kh}^*} \right).
\end{equation}

\[4\] In the case where we are at a refinancing date for the $i$th firm, that is $T_{i,t} = t$, then $S_{i,t} = \max \left( V_{i,t} - F_{i,t}; 0 \right) = V_{i,t} - F_{i,t}$ if we have conditioned upon the survival of the firm. This case is also invertible.
where \( w_{kh}^* \) is an \( m \)-dimensional column vector defined as

\[
    w_{kh}^* = \left( \ln v_{i,kh}^* - \ln v_{i,(k-1)h}^* - \left( \mu_i - \frac{1}{2} \sigma_{V_i} \right) h \right)_{m \times 1}.
\]

C The implicit estimation method

Following Jones, et al (1984) and Ronn and Verma (1984), an equation relating the diffusion coefficient of the stock price process to that of the asset value process can be obtained because stock price is a function of the asset value. Formally, \( S_{i,t} = g_i(V_{i,t}; t, \sigma_{V_i}) \). Applying Itô’s lemma gives rise to

\[
d\ln S_{i,t} = \left( \frac{\mu_i V_{i,t} \Delta_i(t) + \theta_i(t) + \frac{1}{2} \sigma_{V_i}^2 V_i^2(t) \Gamma_i(t) - \frac{\sigma_{V_i}^2 V_i^2(t) \Delta_i^2(t)}{2S_i^2(t)}}{S_{i,t}} \right) dt + \frac{\sigma_{V_i} V_{i,t} \Delta_i(t)}{S_{i,t}} dW_{i,t},
\]

where

\[
    \Delta_i(t) \equiv \frac{\partial g_i(V_{i,t}; t, \sigma_{V_i})}{\partial v} = \Phi(d(V_{i,t}, \sigma_{V_i})),
\]

\[
    \Gamma_i(t) \equiv \frac{\partial^2 g_i(V_{i,t}; t, \sigma_{V_i})}{\partial v^2} = \frac{\phi(d(V_{i,t}, \sigma_{V_i}))}{\sigma_{V_i} V_{i,t} (T_i - t)^{1/2}},
\]

\[
    \theta_i(t) \equiv \frac{\partial g_i(V_{i,t}; t, \sigma_{V_i})}{\partial t}
\]

and \( \phi \) and \( \Phi \) denotes respectively the density function and the cumulative function of a standard normal random variable. The diffusion coefficient of the stock return process can thus be written as:

\[
    \sigma_{S_i}(t) = \frac{\sigma_{V_i} V_{i,t} \Delta_i(t)}{S_{i,t}}.
\]

This coefficient is time dependent and stochastic. Although it is inconsistent with the model, the implicit estimation method assumes that the sample standard deviation of stock returns sampled over a time period prior to time \( t \) is a good estimate of \( \sigma_{S_{i,t}} \). This relationship in conjunction with \( S_{i,t} = g_i(V_{i,t}; t, \sigma_{V_i}) \) can be used to solve for two unknowns - \( V_{i,t} \) and \( \sigma_{V_i} \) - using the observed value for \( S_{i,t} \) and the estimate for \( \sigma_{S_{i,t}} \).

In the multi-firm context, the quadratic variation between two stock returns is

\[
    \sigma_{S_i}(t) \sigma_{S_j}(t) \rho_{ij}
\]
because

\[
\begin{align*}
\langle \ln S_i, \ln S_j \rangle_t &= \frac{\sigma V_i \Delta_i(t) \sigma V_j \Delta_j(t)}{S_i(t) S_j(t)} d \langle W_i, W_j \rangle_t \\
&= \frac{\sigma V_i \Delta_i(t) \sigma V_j \Delta_j(t)}{S_i(t) S_j(t)} \rho_{ij} dt \\
&= \sigma S_i(t) \sigma S_j(t) \rho_{ij} dt.
\end{align*}
\]

Using the sample correlation may yield to an estimate for \( \rho_{ij} \). Note that this estimation procedure is again inconsistent with the model because the quadratic variation process between \( \ln S_i \) and \( \ln S_j \) is also time dependent and stochastic.

References


Table 1: Simulation results for Merton’s model in the case of high leverage-high noise to signal

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<td>0.240</td>
<td>0.493</td>
<td>134.507</td>
<td>141.436</td>
<td>-0.016</td>
<td>-0.122</td>
<td>–</td>
</tr>
<tr>
<td>Std</td>
<td>–</td>
<td>–</td>
<td>0.119</td>
<td>0.120</td>
<td>0.036</td>
<td>1003.317</td>
<td>1022.175</td>
<td>0.153</td>
<td>0.755</td>
<td>–</td>
</tr>
</tbody>
</table>

\( \text{True} \) is the parameter value used in the Monte Carlo simulation; \( \text{Mean}, \text{Median} \) and \( \text{Std} \) are the sample statistics computed with the 5000 estimated parameter values; \( \text{cvr} \) is the coverage rate defined as the percentage of the 5000 parameter estimates for which the true parameter value is contained in the \( \alpha \) confidence interval implied by the asymptotic distribution; \( V_{0,1} = 10000, V_{0,2} = 10000, F_1 = 9000, F_2 = 9000, T = 3.00, t_0 = 2.00, r = 0.05 \) and \( N = 500 \).
To be updated

Table 3

Maximum Likelihood Estimation (Merton’s model)

<table>
<thead>
<tr>
<th>Year</th>
<th>$\hat{\mu}$</th>
<th>$\hat{\sigma}$</th>
<th>$\hat{V}_0$</th>
<th>$\hat{C}_0$</th>
<th>$\hat{P}_0$</th>
<th>$\hat{\rho}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$$000's$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1999</td>
<td>0.011</td>
<td>0.204</td>
<td>2,851,034</td>
<td>2.075e-005</td>
<td>0.001</td>
<td>0.173</td>
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<td></td>
<td>( 0.142)</td>
<td>( 0.002)</td>
<td>( 4)</td>
<td>(2.969e-006)</td>
<td>( 0.002)</td>
<td>( 0.040)</td>
</tr>
<tr>
<td>2000</td>
<td>-0.007</td>
<td>0.233</td>
<td>3,046,716</td>
<td>9.955e-004</td>
<td>0.026</td>
<td>0.157</td>
</tr>
<tr>
<td></td>
<td>( 0.178)</td>
<td>( 0.002)</td>
<td>(140)</td>
<td>(7.912e-005)</td>
<td>( 0.047)</td>
<td>( 0.039)</td>
</tr>
<tr>
<td>2001</td>
<td>-0.040</td>
<td>0.226</td>
<td>2,488,371</td>
<td>6.637e-004</td>
<td>0.026</td>
<td>0.120</td>
</tr>
<tr>
<td></td>
<td>( 0.161)</td>
<td>( 0.002)</td>
<td>( 72)</td>
<td>(5.095e-005)</td>
<td>( 0.043)</td>
<td>( 0.044)</td>
</tr>
</tbody>
</table>

Sears Canada

<table>
<thead>
<tr>
<th>Year</th>
<th>$\hat{\mu}$</th>
<th>$\hat{\sigma}$</th>
<th>$\hat{V}_0$</th>
<th>$\hat{C}_0$</th>
<th>$\hat{P}_0$</th>
<th>$\hat{\rho}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$$000's$</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>1999</td>
<td>0.193</td>
<td>0.190</td>
<td>3,317,074</td>
<td>1.729e-007</td>
<td>0.000</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( 0.134)</td>
<td>( 0.004)</td>
<td>( 0)</td>
<td>(8.080e-008)</td>
<td>( 0.000)</td>
<td></td>
</tr>
<tr>
<td>2000</td>
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<td>0.206</td>
<td>5740209</td>
<td>1.987e-012</td>
<td>0.000</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( 0.147)</td>
<td>( 0.005)</td>
<td>( 0)</td>
<td>(0.000e+000)</td>
<td>( 0.000)</td>
<td></td>
</tr>
<tr>
<td>2001</td>
<td>0.113</td>
<td>0.235</td>
<td>4089644</td>
<td>2.226e-005</td>
<td>0.000</td>
<td></td>
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<tr>
<td></td>
<td>( 0.165)</td>
<td>( 0.004)</td>
<td>( 11)</td>
<td>(6.240e-006)</td>
<td>( 0.000)</td>
<td></td>
</tr>
</tbody>
</table>

The maximum likelihood estimates at the beginning of Year are computed using the previous two years of daily time series data; $r$ is set to the yield to maturity of a representative one year Canadian government bond while the maturity of debt is set equal to 1.0 year at all point in time; For each firm, the face value of debt is set equal to 0.5× long term debt plus short term debt obtained from the yearly annual report.