Valuing Credit Derivatives Using Gaussian Quadrature:

A Stochastic Volatility Framework

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Abstract

This paper proposes semi-closed-form solutions to value derivatives on mean-reverting assets. We consider a very general mean-reverting process for the underlying asset and two stochastic volatility processes: the Square-Root process and the Ornstein-Uhlenbeck process. For both models, we derive semi-closed-form solutions for Characteristic Functions, in which we need to solve simple Ordinary Differential Equations, and then invert them to recover the cumulative probabilities using the Gaussian-Laguerre quadrature rule. As benchmarks, we use our models to value European Call options within Black-Scholes (1973) (represents constant volatility and no mean-reversion), Longstaff-Schwartz (1995) (represents constant volatility and mean-reversion), Heston (1993) and Zhu (2000) (represent stochastic volatility and no mean-reversion) frameworks. These comparisons show that we only need polynomials with small degree for convergence and accuracy. Indeed, when applied to our processes (represent stochastic volatility and mean-reversion), the Gaussian-Laguerre rule is very efficient and very accurate. We also show that the mean-reversion could have a large impact on option prices even though the strength of the reversion is small. As applications, we value credit spread options, caps, floors and swaps.

Keywords: Mean-reversion, Stochastic Volatility, Gaussian Quadrature, Inverse Fourier Transform, Feynman-Kac Theorem, Credit Spread Options, Caps, Floors, Swaps.

JEL Classification: G13, C63
Some assets such as interest rates, credit spreads and some commodities are shown to exhibit mean-reversion feature. Many papers price derivatives on these assets under constant volatility assumption. Cox, Ingersoll and Ross (1985) propose a general equilibrium model and derive a square-root interest rate model and the discount bond pricing formula as well as the bond option price. Longstaff and Schwartz (1995) show that the log-credit spreads are mean-reverting. Assuming that they could be modelled with a Gaussian process, they derive a simple closed-form solution for European options. Schwartz (1997) proposes a mean-reverting process for commodities in order to price some derivatives. But all these models have the common point that the volatility is assumed non-stochastic, which is a strong simplification that omits some empirical features like leptokurtic distributions and leads to unrealistic option prices.

In this way, substantial progress has been made in developing more realistic option pricing models by incorporating stochastic volatility and jumps. Heston (1993) prices European options on stocks, bonds and currencies under a square-root volatility process. In the same way, Bakshi, Cao and Chen (1997) combine stochastic volatility and jumps to test the empirical performance of some alternative option pricing models. Schöbel and Zhu (1998) and Zhu (2000) propose a more elegant method to derive option prices under some volatility models such as the square-root and the Ornstein-Uhlenbeck. While all these papers propose simple and easy-to-use closed-form solutions to non-mean-reverting assets derivatives under stochastic volatility assumption, there is a little literature for mean-reverting assets within stochastic volatility frameworks.

Assuming a square-root volatility and Vasicek’s (1977) interest rate process, Fong and Vasicek (1992) develop the fundamental partial differential equation for interest rate contingent claims but derive a closed-form solution only for discount bonds. This solution
requires a heavy computation of the confluent hypergeometric function within the complex numbers algebra. To override this difficulty, Selby and Strickland (1995) propose a series solution for the discount bond price that is very efficient. In another paper (1997), they also develop a Monte Carlo valuation of other interest rate derivatives under the Fong and Vasicek (1992) model. In a discrete-time framework, Tahani (2000) proposes a closed-form valuation formula for credit spread options under GARCH as a generalization of Longstaff and Schwartz (1995) and Heston and Nandi (2000) models. He also shows that the GARCH used has the square-root mean-reverting process, one of the two models used in the current paper, as a continuous-time limit.

In this paper, we propose to derive pricing formulas for options on mean-reverting assets within two stochastic volatility frameworks, the square-root and the Ornstein-Uhlenbeck processes. In this, we generalize the Longstaff and Schwartz (1995) constant volatility model and the work done by Heston (1993) and Zhu (2000) by incorporating a mean-reverting component. Our work also extends the Fong and Vasicek (1992) model since we propose a semi-analytic valuation framework for some derivatives on general mean-reverting assets instead of using Monte Carlo simulation as done by Selby and Strickland (1997). Monte Carlo methods may need a large number of paths simulation which makes semi-analytic valuation, when possible, much more efficient.

For both stochastic volatility models considered in this paper, we derive semi-closed-form characteristic functions in which we only need to solve simple ordinary differential equations (ODEs). In the square-root case, even though we derive complete closed-form solution (it involves the Whittaker functions that are related to the confluent hypergeometric function), we show that a numerical resolution of the ODEs provides us with as accurate values as the exact ones but in much less time. This is due to the fact that when one deals
with complex functions that could only be computed approximately as a series expansion (like Whittaker and the confluent hypergeometric functions) even with some mathematical softwares such as Maple® or Mathematica®, one has to face large time computation and usually overflow errors.

Once the characteristic function derived in a semi-closed-form way, we use the inverse Fourier transform technique to get the associated cumulative probabilities by a numerical integration based on the Gauss-Laguerre quadrature rule. The Gaussian integration technique was proved very efficient and accurate in many papers among which Bates (1996), who prices currency options within a stochastic volatility and jumps framework, and Sullivan (2000) who proposes an approximation to American Put options. In our frameworks, using some benchmark such as the Black and Scholes (1973), the Longstaff and Schwartz (1995), the Heston (1993) and the Zhu (2000) models, the Gauss-Laguerre quadrature rule is shown to be very accurate and convergent to the true price even with small polynomial degree. As applications, we value credit spread options, caps, floors and swaps.

The contribution of this work is twofold. We propose a semi-analytic procedure to price derivatives on very general mean-reverting underlying assets under stochastic volatility assumption. We also show that a small mean-reversion coefficient (e.g. for credit spreads, it is of order 0.02) could have a large impact on option prices (up to 20% - 40%).

The next section presents a general mean-reverting framework and how to compute the characteristic function. Section II derives semi-closed-form solutions for characteristic functions under both the square-root and the Ornstein-Uhlenbeck volatility assumptions. Section III presents the numerical integration procedure using the Gaussian quadrature rules
to recover the cumulative probabilities. Section IV values some credit spread derivatives and their Greeks as particular applications. Section V presents some results on convergence and efficiency. Section VI will conclude.
I General mean-reverting framework and characteristic functions

We consider a more general model for the underlying asset given under the historical measure $P$ by:

$$dX_t = (\mu - \alpha X_t)dt + b(V_t)dz_t(t)$$  \hspace{1cm} (1)

where $X$ could be denoting a log-stock price or a log-currency (if $\alpha = 0$), or any general mean-reverting process like log-credit spreads as in Longstaff and Schwartz (1995) and Tahani (2000). For the volatility, we consider the following general diffusion:

$$d(a(V_t)) = \kappa(\theta - a(V_t))dt + b'(V_t)dZ_2(t)$$  \hspace{1cm} (2)

where $a(\cdot)$, $b(\cdot)$ and $b'(\cdot)$ are real-valued functions of the squared volatility $V$ and will be specified later. The parameters $\mu$, $\alpha$, $\kappa$ and $\theta$ are constant. $Z_1$ and $Z_2$ are correlated Brownian motions under the historical measure $P$.

We assume that the volatility risk-premium is proportional to $a(V)$ as in Heston (1993) and that the risk-premium for the underlying asset is proportional to $b^2(V)$ such that the two diffusions under a risk-neutral measure $Q$ become:

$$dX_t = [\mu - \alpha X_t - \gamma b^2(V_t)]dt + b(V_t)dW_1(t)$$  \hspace{1cm} (3)

$$d(a(V_t)) = [\kappa \theta - (\kappa + \pi) a(V_t)]dt + b'(V_t)dW_2(t)$$  \hspace{1cm} (4)

where $\gamma$ and $\pi$ denote the unit risk-premiums and $W_1$ and $W_2$ are correlated Brownian motions under $Q$. To value option-like derivatives, we must compute the following types of expectations under $Q$ at time $t$:

$$E_t^Q \left( \exp \left( - \int_t^T r(s)ds \right) \times e^{X_T} \times 1_{X_T > \ln(K)} \right)$$  \hspace{1cm} (5)

and

$$E_t^Q \left( \exp \left( - \int_t^T r(s)ds \right) \times 1_{X_T > \ln(K)} \right)$$  \hspace{1cm} (6)
In order to obtain simpler expressions for these expectations, we consider two probability measures $Q_1$ and $Q_2$ equivalent to $Q$ and defined by their Radon-Nikodym derivatives:

$$
\frac{dQ_1}{dQ} = g_1(t, T) = \frac{\exp\left[-\int_t^T r(s)ds \right] \times e^{X_T}}{E_t^Q\left(\exp\left[-\int_t^T r(s)ds \right] \times e^{X_T}\right)}
$$

(7)

$$
\frac{dQ_2}{dQ} = g_2(t, T) = \frac{\exp\left[-\int_t^T r(s)ds \right]}{E_t^Q\left(\exp\left[-\int_t^T r(s)ds \right]\right)}
$$

(8)

$Q_2$ is simply the so-called $T$-forward measure. Equations (5) and (7) give:

$$
E_t^Q\left(\exp\left[-\int_t^T r(s)ds \right] \times e^{X_T} \times 1_{X_T > \ln(K)}\right) = Q_1(X_T > \ln(K)) \times E_t^Q\left(e^{-\int_t^T r(s)ds} \times e^{X_T}\right)
$$

(9)

and Equations (6) and (8) give:

$$
E_t^Q\left(\exp\left[-\int_t^T r(s)ds \right] \times 1_{X_T > \ln(K)}\right) = Q_2(X_T > \ln(K)) \times P(t, T)
$$

(10)

where $P(t, T)$ denotes the zero-coupon bond maturing at $T$. We also define the characteristic functions of the process $X$ under $Q_1$ and $Q_2$ by:

$$
f_j(\phi) \equiv E_t^Q, \left[\exp(i\phi X_T)\right] \quad \text{for } j = 1, 2
$$

(11)

Expressed under the risk-neutral measure $Q$, the characteristic function $f_1$ becomes:

$$
f_1(\phi) \equiv E_t^Q, \left[g_1(t, T) \exp(i\phi X_T)\right]
$$

$$
= \frac{E_t^Q\left[\exp\left(-\int_t^T r(s)ds\right) \times \exp((1 + i\phi)X_T)\right]}{E_t^Q\left(\exp\left[-\int_t^T r(s)ds \right] \times e^{X_T}\right)}
$$

(12)
and $f_2$ becomes:

$$f_2(\phi) = E_t^Q \left[ g_2(t, T) \exp(i\phi X_T) \right]$$

$$E_t^Q \left[ \exp \left( - \int_t^T r(s) ds \right) \times \exp \left( i\phi X_T \right) \right]$$

$$E_t^Q \left[ \exp \left( - \int_t^T r(s) ds \right) \right]$$

(13)

These expressions will be derived dependently on the risk-free rate specification. However, if we define an “actualized characteristic function” $f$ by:

$$f(\psi) = E_t^Q \left[ \exp \left( - \int_t^T r(s) ds \right) \exp(\psi X_T) \right]$$

(14)

we can simplify Equations (12) and (13) as:

$$f_1(\phi) = \frac{f(1 + i\phi)}{f(1)}$$

(15)

$$f_2(\phi) = \frac{f(i\phi)}{f(0)}$$

(16)

Under our stochastic volatility and mean-reverting models, we will show that these characteristic functions can be expressed as log-linear combinations of some functions that solve simple ODEs.

To recover the cumulative probabilities in Equations (9) and (10), we apply the Fourier inversion theorem (see Kendall and Stuart 1977) to obtain:

$$Q_1(X_T > \ln(K)) = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \text{Re} \left( \frac{f(1 + i\phi)}{i\phi f(1)} K^{-i\phi} \right) d\phi$$

(17)

$$Q_2(X_T > \ln(K)) = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \text{Re} \left( \frac{f(i\phi)}{i\phi f(0)} K^{-i\phi} \right) d\phi$$

(18)
where \( \text{Re}(\cdot) \) denotes the real part of a complex number. These integrals are well-defined (see Appendix C) and convergent. Although they cannot be computed analytically, we will use numerical techniques such as Gaussian quadrature to do it. The next section will consider two different stochastic volatility models by choosing appropriate \( a(V) \) and \( b(V) \) functions and derive their characteristic functions.

II  Stochastic volatility models

II.1 Square-root mean-reverting model

In this subsection, we generalize Heston (1993) model by incorporating a mean-reverting underlying asset. The model is given under the risk-neutral measure \( Q \) by:

\[
dX_t = \left( \mu - \alpha X_t - \gamma V_t \right) dt + \sqrt{V_t} dW_1(t)
\]

\[
dV_t = \left( \kappa \theta - \lambda V_t \right) dt + \sigma \sqrt{V_t} dW_2(t)
\]

where \( d\langle W_1, W_2 \rangle_t = \rho dt \). For all models, we assume a constant risk-free rate denoted by \( r \).

The characteristic function can be expressed by (for details, see Appendix A.1):

\[
E_t^Q \left( e^{i\tau X_t} \right) = \exp \left( \psi \left( \frac{1}{\alpha} \left( -\alpha \tau \right)^2 - \frac{1}{2} \left( 1 - e^{-\alpha (T-t)} \right) \right) - \frac{\rho \kappa \theta}{\alpha} \psi \left( 1 - e^{-\alpha (T-t)} \right) \right)
\]

\[
\times E_t^Q \left( e^{\int_{T}^{T} \left[ \epsilon_1 (T-s) V_s ds \right]} \right)
\]

where

\[
\begin{align*}
\epsilon_1 (\tau) &= \left( \frac{\rho}{\sigma} (\alpha - \lambda) + \gamma \right) \psi (-\alpha \tau) - \frac{1}{2} \psi^2 (1 - \rho^2) \exp (-2 \alpha \tau) \\
\epsilon_2 &= \frac{\rho}{\sigma} \psi
\end{align*}
\]
Putting $\alpha = 0$ leads obviously to Zhu (2000) equations. The expectation term will be computed using Feynman-Kac theorem as given in Karatzas and Shreve (1991) (see Appendix D). If we define the function $F(t,V)$ as:

$$F(t,V) = \mathbb{E}^Q_{t,T} \left[ \exp \{ \varepsilon_2 V_T \} \exp \left\{ - \int_t^T \varepsilon_1 (T-s) V_s ds \right\} \right]$$

(23)

then by Feynman-Kac theorem, we have that $F(t,V)$ must satisfy the following partial differential equation (PDE):

$$\begin{align*}
\frac{\partial F}{\partial t} + \frac{1}{2} \sigma^2 V \frac{\partial^2 F}{\partial V^2} + (\kappa \theta - \lambda V) \frac{\partial F}{\partial V} - \varepsilon_1 (T-t) V F &= 0 \\
F(T,V) &= \exp(\varepsilon_2 V)
\end{align*}$$

(24)

And if we assume that $F(t,V)$ is log-linear and given by:

$$F(t,V) = \exp[D(T-t)V + C(T-t)]$$

(25)

then $D(\tau)$ and $C(\tau)$ satisfy the following ODEs:

$$\begin{align*}
D'(\tau) - \frac{1}{2} \sigma^2 D^2(\tau) + \lambda D(\tau) + \varepsilon_1(\tau) &= 0 \\
D(0) &= \varepsilon_2 = \frac{\rho \psi}{\sigma}
\end{align*}$$

(26)

and

$$\begin{align*}
C'(\tau) - \kappa \theta D(\tau) &= 0 \\
C(0) &= 0
\end{align*}$$

(27)

Solving these ODEs will give the unique solution to the PDE in Equation (24) and then to the actualized characteristic function $f(\psi)$. Although the ODE (26) is of Riccati type, we don’t have simple analytic solutions like in Heston (1993) and Zhu (2000). Because of the
mean-reverting feature, the function $\varepsilon_1(\tau)$ is of exponential type while it’s independent of the time variable $\tau$ in Zhu (2000). However, these first-degree ODEs can be solved easily using numerical methods such as Runge-Kutta formula or Adams-Bashforth-Moulton method. For details about these methods, see Dormand and Prince (1980), Shampine (1994) or Shampine and Gordon (1975).

The ODEs (26-27) have analytic solutions (given in Appendix A.2) that involve the Whittaker functions. They need much more time (about 10 times on average) to be computed for large values of $\phi$ (or equivalently $\psi$ as defined in Equations 14-16) than solved numerically for the same order of accuracy (the relative error is about $10^{-8}$). This may be due to the fact that the confluent hypergeometric function is approximated by a series expansion in all mathematical softwares and the calculation could be slow for large input values (one could also face overflow errors as well). The series expansion for the Fong and Vasicek (1992) discount bond price in Selby and Strickland (1995) is efficient because $\phi$ is always equal to 1.

The actualized characteristic function $f(\psi)$ for the square-root mean-reverting model is then given by:

$$
\begin{align*}
f(\psi) &\equiv \mathbb{E}_t^Q \left[ \exp \left( - \int_t^T r(s) ds \right) \exp(\psi X_t) \right] \\
&= e^{-\tau(t-T)} \times \exp \left( \psi e^{-\alpha(T-t)} X_t + \frac{\mu}{\alpha} \psi \left( 1 - e^{-\alpha(T-t)} \right) - \frac{\rho}{\sigma} \frac{\kappa \theta}{\alpha} \psi \left( 1 - e^{-\alpha(T-t)} \right) - \frac{\rho}{\sigma} \psi e^{-\alpha(T-t)} V_t \right) \\
&\times \exp \left( D(T-t; \psi)V_t + C(T-t; \psi) \right)
\end{align*}
$$

Now that we can evaluate numerically expressions like $f(i\phi)$ and $f(1+i\phi)$ for all possible values of $\phi$, we are able to compute cumulative probabilities under measures $Q_1$. 

11
and $Q_2$ by inverting the Fourier transforms which leads to evaluate the integrals given in Equations (17) and (18). This will be done by the Gaussian quadrature rule using Laguerre polynomials. Section III will describe this method and shows how it applies to our models.

II.2 Ornstein-Uhlenbeck mean-reverting model

A. Risk-premium for the underlying asset is proportional to the squared volatility

In this subsection, we use an Ornstein-Uhlenbeck model for the volatility and a mean-reverting underlying asset. The model is given under the risk-neutral measure $Q$ by:

\[ dX_t = (\mu - \alpha X_t - \gamma \sigma_t^2)dt + \sigma_t dW_1(t) \]  
\[ d\sigma_t = (\kappa \theta - \lambda \sigma_t)dt + \beta dW_2(t) \]

where $d\langle W_1, W_2 \rangle = \rho dt$. We can write the characteristic function as (for details, see Appendix B):

\[ E_t^Q(e^{\kappa X_T}) = \exp \left\{ \frac{\psi}{\alpha} e^{-\alpha(T-t)} X_t + \frac{\mu}{\alpha} \psi(1 - e^{-\alpha(T-t)}) - \frac{\rho \beta}{2\alpha} \psi(1 - e^{-\alpha(T-t)}) \right\} \]

\[ \times \left[ \frac{\rho \psi e^{-\alpha(T-t)}}{2\beta} \sigma_t^2 \right] \]

\[ \times \left[ \exp \left( \eta_1 \sigma_T^2 - \int_t^T \eta_2 (T-s) \sigma_s ds - \int_t^T \eta_3 (T-s) \sigma_s^2 ds \right) \right] \]

where

\[ \eta_1(\tau) = \left( \frac{\alpha \rho}{2\beta} - \frac{\rho \lambda}{\beta} + \gamma \right) \psi \exp(-\alpha \tau) - \frac{1}{2} \psi^2 \left( 1 - \rho^2 \right) \exp(-2\alpha \tau) \]

\[ \eta_2(\tau) = \frac{\rho \kappa \theta}{\beta} \psi \exp(-\alpha \tau) \]

\[ \eta_3 = \frac{\rho}{2\beta} \psi \]

Again, putting $\alpha = 0$ leads to Zhu (2000) equations. Define the function $G(t, \sigma)$ as:
\[ G(t, \sigma) = E^0_t \left\{ \exp \left( \eta_3 \sigma T - \int_t^T \eta_2 (T - s) \sigma_s ds - \int_t^T \eta_1 (T - s) \sigma_s^2 ds \right) \right\} \]  

(33)

By Feynman-Kac theorem, we have that \( G(t, \nu) \) must satisfy the following PDE:

\[
\begin{aligned}
\frac{\partial G}{\partial t} + \frac{1}{2} \beta^2 \frac{\partial^2 G}{\partial \sigma^2} + (\kappa \theta - \lambda \sigma) \frac{\partial G}{\partial \sigma} - \left( \eta_1 (T - t) \sigma^2 + \eta_2 (T - t) \sigma \right) G = 0
\end{aligned}
\]  

(34)

Assume that \( G(t, \sigma) \) is log-linear and given by:

\[ G(t, \sigma) = \exp \left[ \frac{1}{2} E(T - t) \sigma^2 + D(T - t) \sigma + C(T - t) \right] \]  

(35)

then \( E(\tau), D(\tau) \) and \( C(\tau) \) satisfy the following ODEs:

\[
\begin{aligned}
\frac{1}{2} E'(\tau) - \frac{1}{2} \beta^2 E^2(\tau) + \lambda E(\tau) + \eta_1(\tau) &= 0 \\
E(0) &= 2 \eta_3 = \frac{\rho}{\beta} \psi
\end{aligned}
\]  

(36)

\[
\begin{aligned}
D'(\tau) - \beta^2 E(\tau) D(\tau) + \lambda D(\tau) - \kappa \theta E(\tau) + \eta_2(\tau) &= 0 \\
D(0) &= 0
\end{aligned}
\]  

(37)

and

\[
\begin{aligned}
C'(\tau) - \frac{1}{2} \beta^2 E(\tau) - \frac{1}{2} \beta^2 D^2(\tau) - \kappa \theta D(\tau) &= 0 \\
C(0) &= 0
\end{aligned}
\]  

(38)

Although the Riccati-type ODE (36) has an exact analytic solution (see Appendix B.2), the ODEs (37-38) do not have closed-form solutions. As discussed earlier, all these ODEs will
be solved numerically. The actualized characteristic function \( f(\psi) \) for the Ornstein-Uhlenbeck mean-reverting model is then given by:

\[
f(\psi) \equiv E_t^Q \left[ \exp \left( -\int_t^T r(s) \, ds \right) \exp(\psi X_T) \right]
\]

\[
= e^{-r(T-t)} \exp \left( \psi e^{-\alpha(T-t)} X_t + \frac{\mu}{\alpha} \psi (1 - e^{-\alpha(T-t)}) - \frac{\rho \beta}{2\alpha} \psi (1 - e^{-\alpha(T-t)}) \right)
\]

\[
= e^{-r(T-t)} \exp \left( -\frac{\rho}{2\beta} \psi e^{-\alpha(T-t)} \sigma_t^2 \right)
\]

\[
\times \exp \left[ \frac{1}{2} E(T-t; \psi) \sigma_t^2 + D(T-t; \psi) \sigma_t + C(T-t; \psi) \right]
\]

(39)

**B. Risk-premium for the underlying asset is proportional to the volatility**

If the risk-premium for the underlying asset \( X \) is proportional to the volatility instead of its square, the model’s equations become:

\[
dX_t = (\mu - \alpha X_t - \gamma \sigma_t) \, dt + \sigma_t \, dW_1(t)
\]

(40)

\[
d\sigma_t = (\kappa \theta - \lambda \sigma_t) \, dt + \beta \, dW_2(t)
\]

(41)

and for the characteristic function, we obtain with the same calculations as before:

\[
E_t^Q \left( e^{\psi X_T} \right) = \exp \left( \psi e^{-\alpha(T-t)} X_t + \frac{\mu}{\alpha} \psi (1 - e^{-\alpha(T-t)}) - \frac{\rho \beta}{2\alpha} \psi (1 - e^{-\alpha(T-t)}) \right)
\]

\[
- \frac{\rho}{2\beta} \psi e^{-\alpha(T-t)} \sigma_t^2
\]

\[
\times E_t^Q \left( \exp \left( \omega \sigma_t^2 - \int_t^T \omega_2(T-s) \sigma_s \, ds - \int_t^T \omega_1(T-s) \sigma_s^2 \, ds \right) \right)
\]

(42)

where
Using Feynman-Kac theorem as before gives for the actualized characteristic function:

\[
\begin{aligned}
\int_{0}^{\tau} \left[ \exp(-\alpha\tau) - \frac{1}{2} \psi^2(1 - \rho^2) \exp(-2\alpha\tau) \right] \\
\left( \frac{\rho \kappa \theta}{\beta} + \gamma \right) \psi \exp(-\alpha\tau) \\
\frac{\rho}{2\beta} \psi
\end{aligned}
\]

(43)

III Numerical integration using Gaussian quadrature

In general, a quadrature rule approximation allows to estimate an integral of a function, \( g(\phi) \), over a given interval with a linear combination of function values in the interval \([a, b]\). After specifying a set of abscissas \( \phi_{j} \) and their corresponding weights \( \omega_{j} \), the integral is approximated by:

\[
\int_{a}^{b} w(\phi) g(\phi) d\phi \equiv \sum_{j=1}^{n} \omega_{j} g(\phi_{j})
\]

(45)

where \( w \) is a weight function to be specified dependently on the rule which is used. The abscissas and the weights are specified such that this approximation is exact for any given
polynomial function with a maximum degree. The highest degree is called the order of the quadrature rule. While rules such as Trapezoidal and Simpson’s specify a set of equally spaced abscissas and choose the weights to maximize the order, Gaussian rules determine both abscissas and weights to maximize the order. For \( n \) abscissas and \( n \) weights, the highest order is \( 2n - 1 \). Furthermore, in many studies, Gaussian rules are shown to converge faster than the classic Trapezoidal and Simpson’s rules and give greater accuracy even for small \( n \) (Sullivan 2000).

The Gauss-Laguerre quadrature rule over the interval \([0, +\infty]\) has the following weight function:

\[
w(\phi) = \exp(-\phi)\]

while the abscissas and the weights solve the following \( 2n \) equations:

\[
\int_0^{+\infty} \exp(-\phi) \phi^q d\phi = \omega_1 \exp(-\phi_1) \phi_1^q + \omega_2 \exp(-\phi_2) \phi_2^q + \cdots + \omega_n \exp(-\phi_n) \phi_n^q
\]

for \( q = 0, \ldots, 2n - 1 \). These abscissas and weights can also be determined using some properties of Laguerre polynomials. They are tabulated in Abramowitz and Stegun (1968).

The next subsection gives a brief overview of these polynomials and shows how to specify the rule. The one after will apply this quadrature rule to invert our models characteristic functions to recover cumulative probabilities.

### III.1 A brief overview of Laguerre polynomials

The \( n \)-th Laguerre polynomial is defined by:

\[
L_n(\phi) = \frac{1}{n!} \exp(\phi) \times \frac{d^n}{d\phi^n} \left[ \exp(-\phi) \phi^n \right] = \frac{(-1)^n}{n!} \phi^n + \cdots + \frac{1}{2!} \phi^2 - \phi + 1
\]

(47)
They have many characteristics, among which the “orthonormality” with respect to the weight function:

\[
\int_0^{+\infty} \exp(-\phi) L_n(\phi) L_p(\phi) d\phi = \delta_{np} = \begin{cases} 
0 & \text{if } n \neq p \\
1 & \text{if } n = p
\end{cases}
\] (48)

where \( \delta_{np} \) is the Kronecker’s symbol. We can also proof that the \( n \)-th Lageurre polynomial has exactly \( n \) real zeros over the interval \([0, +\infty[\). These zeros are the abscissas, \( (\phi_j)_{j=1,...,n} \), needed for the Gauss-Laguerre quadrature rule of order \( n \). The associated weights, \( (\omega_j)_{j=1,...,n} \), are then given by:

\[
\omega_j = \frac{1}{n^2} \frac{\phi_j}{L_{n-1}(\phi_j)}^{n} \quad j = 1, ..., n
\] (49)

In order to apply this integration method to our models, we need to modify the weights to take into account the function to be integrated. Recall that we must value this type of integrals:

\[
\int_0^{+\infty} g(\phi) d\phi = \int_0^{+\infty} \exp(-\phi) [\exp(\phi) g(\phi)] d\phi = \sum_{j=1}^{n} \omega_j \exp(\phi_j) g(\phi_j)
\] (50)

While \( n \) increases, the sum converges to the true value if the function \( \exp(\phi) g(\phi) \) satisfies some assumptions as discussed in Davis and Rabinowitz (1984).

**III.2 Recovering cumulative probabilities**

For a fixed order \( n \), we setup the abscissas \( (\phi_j)_{j=1,...,n} \) and the modified weights \( (\omega_j \exp(\phi_j))_{j=1,...,n} \) as shown before. We also choose all the model’s parameters as well as
the time $t$, the maturity $T$, the strike $K$, the initial value for the underlying asset $X$, and, depending on the model, the initial value for the volatility $\sigma$, or the squared volatility $V$. For every abscissa $\phi_j$ or equivalently $\psi_j$ as defined earlier by $\psi_j = 1 + i\phi_j$ or by $\psi_j = i\phi_j$, we solve the ODEs in Equations (26-27) for the square-root model and Equations (36-38) for the Ornstein-Uhlenbeck model to get the functions values $E(T - t; \psi_j)$, $D(T - t; \psi_j)$ and $C(T - t; \psi_j)$ defined in section II for every $j = 1, ..., n$. We can then compute the actualized characteristic functions values $f(\psi_j)$ at the needed points $\psi_j$ and the cumulative probabilities can then be approximated by:

$$Q_1(X_T > \ln(K)) = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \Re \left( \frac{f(1 + i\phi)}{i\phi f(1)} K^{-i\phi} \right) d\phi$$

$$= \frac{1}{2} + \frac{1}{\pi} \sum_{j=1}^{n} \omega_j \exp(\phi_j) \Re \left( \frac{f(1 + i\phi_j)}{i\phi_j f(1)} \exp\{-i\phi_j \ln(K)\} \right)$$

(51)

and

$$Q_2(X_T > \ln(K)) = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \Re \left( \frac{f(i\phi)}{i\phi f(0)} K^{-i\phi} \right) d\phi$$

$$= \frac{1}{2} + \frac{1}{\pi} \sum_{j=1}^{n} \omega_j \exp(\phi_j) \Re \left( \frac{f(i\phi_j)}{i\phi_j f(0)} \exp\{-i\phi_j \ln(K)\} \right)$$

(52)

Theoretically, as $n$ becomes large, these approximations converge to the true probability values. As it will be shown with many valuation examples, a fast convergence and a good accuracy can be achieved even with small $n$. 


IV Credit spread options, caps, floors and swaps valuation

IV.1 Credit spread options

A credit spread option gives the right to buy or sell the credit at the strike price until or at the expiration date dependently if the option is American or European. One could buy a credit spread option for hedging its credit risk exposures against up or down movements in a credit value as well as for speculative purposes. For an exhaustive credit derivatives overview, see Howard (1995).

More specifically, denoting the maturity date by $T$ and the strike by $K$, under the models studied in earlier sections, the European Call premium is given by:

$$\text{Call}(t, T) = f(t, T; 1) Q_1^{i, T} (X_T > \ln(K)) - f(t, T; 0) K Q_2^{i, T} (X_T > \ln(K))$$

(53)

where the actualized characteristic function $f(t, T; \psi)$ and the cumulative probabilities are defined as before by:

$$f(t, T; \psi) = E_t^Q \left[ \exp \left( - \int_t^T r(s) ds \right) \exp(\psi X_T) \right]$$

(54)

and

$$Q_1^{i, T} (X_T > \ln(K)) = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \text{Re} \left( \frac{f(t, T; 1 + i\phi)}{i\phi f(t, T; 1)} K^{-i\phi} \right) d\phi$$

(55)

$$Q_2^{i, T} (X_T > \ln(K)) = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \text{Re} \left( \frac{f(t, T; i\phi)}{i\phi f(t, T; 0)} K^{-i\phi} \right) d\phi$$

(56)

The European Put can be derived using the Call-Put parity:

$$\text{Put}(t, T) = f(t, T; 0) K Q_2^{i, T} (X_T < \ln(K)) - f(t, T; 1) Q_1^{i, T} (X_T < \ln(K))$$

(57)
In order to hedge options against changes in the underlying asset and in the volatility, we need to derive the Greeks. The calculus details are given in Appendix E. For both stochastic volatility models, the Delta is given by:

\[
\Delta(t,T) \equiv \frac{\partial \text{Call}(t,T)}{\partial e^{X_t}} = e^{-\alpha(T-t)} e^{-X_t} f(t,T;1) Q_1^{1,T} (X_T > \ln(K)) \tag{58}
\]

and the Gamma by:

\[
\Gamma(t,T) \equiv \frac{\partial \Delta(t,T)}{\partial e^{X_t}} = \left(1 - e^{-\alpha(T-t)}\right) e^{-X_t} \Delta(t,T) + \frac{e^{-2\alpha(T-t)} e^{-2X_t}}{\pi} \int_{0}^{+\infty} \text{Re} \left( f(t,T;1+i\phi) K^{-i\phi} \right) d\phi \tag{59}
\]

The derivation of the Vega depends on the stochastic volatility model used. For the square-root model, the Vega is given by:

\[
\text{ Vega}(t,T) \equiv \frac{\partial \text{Call}(t,T)}{\partial V_t} = \left(D(T-t;1) - \frac{\rho}{\sigma} e^{-\alpha(T-t)}\right) f(t,T;1) Q_1^{1,T} (X_T > \ln(K))
+ \frac{1}{\pi} \int_{0}^{+\infty} \text{Re} \left( \frac{D(T-t;1+i\phi) - D(T-t;1)}{i\phi} f(t,T;1+i\phi) K^{-i\phi} \right) d\phi
- \frac{K}{\pi} \int_{0}^{+\infty} \text{Re} \left( \frac{D(T-t;1+i\phi)}{i\phi} f(t,T;1+i\phi) K^{-i\phi} \right) d\phi \tag{60}
\]

and for the Ornstein-Uhlenbeck model by:
\[Vega(t, T) \equiv \frac{\partial \text{Call}(t, T)}{\partial \sigma_i}\]

\[= \left( E(T - t; 1)\sigma_i + D(T - t; 1) - \frac{\partial}{\partial \sigma_i} e^{-\sigma_i(T-t)}\sigma_i \right) f(t, T; 1) Q_{1,T} (X_T > \ln(K))\]

\[+ \frac{1}{\pi} \int_0^{+\infty} \text{Re} \left\{ \frac{E(T - t; 1 + i\phi) - E(T - t; 1)}{i\phi} \sigma_i \right\} f(t, T; 1 + i\phi) K^{-i\phi} d\phi\]

\[+ \frac{K}{\pi} \int_0^{+\infty} \text{Re} \left\{ \frac{E(T - t; i\phi)}{i\phi} \sigma_i + \frac{D(T - t; i\phi)}{i\phi} \right\} f(t, T; i\phi) K^{-i\phi} d\phi\]  \hspace{1cm} (61)

**IV.2 Credit spread Cap and Floor**

A credit spread cap or floor provides the right to get payoffs at periodic dates called the *reset dates*. At each reset date, the cap/floor payoff is the same as for a call/put. In this, the cap/floor can be seen as a sequence of many calls/puts called *caplets* or *floorlets*. The Figure below shows the reset dates and the associated payoffs for a credit spread cap with a maturity \(T\) and different strike prices corresponding to the \(n\) periods.

The cap/floor premium is then equal to the sum of the corresponding caplets/floorlets premia. The cap premium is given by:

\[\text{Cap}(t) = \sum_{j=1}^{n} f(t, t_j; 1) Q_{1}^j \left(X(t_j) > \ln(K_j)\right)\]

\[\quad - \sum_{j=1}^{n} f(t, t_j; 0) K_j Q_{2}^j \left(X(t_j) > \ln(K_j)\right)\]  \hspace{1cm} (62)
where \( f(t, t_j; \psi), \ Q^j_1 \left( X(t_j) > \ln(K_j) \right) \) and \( Q^j_2 \left( X(t_j) > \ln(K_j) \right) \) are defined for each reset date as for the Call in Equations (54-56). The floor premium can also be valued by:

\[
Floor(t) = \sum_{j=1}^{n} f(t, t_j; 0) K_j Q^j_2 \left( X(t_j) < \ln(K_j) \right) - \sum_{j=1}^{n} f(t, t_j; 1) Q^j_1 \left( X(t_j) < \ln(K_j) \right)
\]  

(63)

as well as the Delta by:

\[
\frac{\partial Cap(t)}{\partial e^{X_i}} = e^{-x_i} \times \sum_{j=1}^{n} e^{-\alpha(t_i-t)} f(t, t_j; 1) Q^j_1 \left( X(t_j) > \ln(K_j) \right)
\]

(64)

**IV.3 Credit spread Swap**

A credit spread swap is an obligation to get payoffs at periodic dates called the reset dates. At each reset date, the swap payoff is the same as for a forward contract. In this, the swap can be seen as a sequence of many forward contracts called swaplets. The Figure below shows the reset dates and the associated payoffs for a credit spread swap with a maturity \( T \) and a strike price \( K \).

The swap value is then equal to the difference between the cap and the floor values with the same strike price at the reset dates. By analogy with interest rate swaps, we can derive the value of the strike price which makes the swap value at the beginning equal to 0:

\[
\text{Swap}(t)
\]
\[ K = \frac{\sum_{j=1}^{n} f(t, t_j; 1)}{\sum_{j=1}^{n} f(t, t_j; 0)} \]  

where \( f(t, t_j; \psi) \) are defined for each reset date as in Equation (54).
V Empirical results on convergence and the impact of mean-reversion

To assess the accuracy and the efficiency of our procedure, we priced many European call options within different frameworks, Black and Scholes (1973) (B&S hereafter), Longstaff and Schwartz (1995) (L&S hereafter) and Zhu (2000) (the Heston’s (1993) model is one of the models studied by Zhu). For different parameters, we try to converge to the exact true price. The true price for B&S and L&S frameworks are given by the corresponding simple formula, while for Zhu framework, we use a Matlab® routine for numerical integration.

The Tables 1 to 4 present the results for different maturities, different strikes and different models parameters. All these applications show that a good accuracy is achieved even with small quadrature rule order, between 10 and 15 depending on which model is used and within which framework. We also find that the relative pricing errors are very small and converge to 0. The Figures 1 to 4 present the relative errors. Notice that for Zhu (2000) framework, the square-root model converges faster than the Ornstein-Uhlenbeck model.

The efficiency and the accuracy of our semi-analytic procedure within these “exact” frameworks are still true for our mean-reverting frameworks. Indeed for both the square-root and the Ornstein-Uhlenbeck models, the “true” asymptotic price (this asymptotic price could be computed with a Monte Carlo simulation) is attained for small quadrature rule orders between 12 and 15. The Tables 5 and 6 present the convergence and the Figures 5 and 6 present the relative errors with respect to the asymptotic price. Again, the convergence is faster for the square-root model than for the Ornstein-Uhlenbeck model.

We also show that the mean-reversion could have a large impact on option prices and that is true even though the strength of the reversion is small. The results presented in Tables
7 and 8 show that with a small mean-reversion coefficient (for credit spreads, \( \alpha \) is about 0.02), the relative impact with respect to the “no mean-reversion option price” (i.e. \( \alpha = 0 \)) is between 20\% and 40\% depending on the maturity.
VI Conclusion

In this paper, we propose semi-analytic pricing formulas for derivatives on mean-reverting assets within two stochastic volatility frameworks. In this, we generalize Longstaff and Schwartz (1995) by making the volatility stochastic; and Heston (1993) and Zhu (2000) models by incorporating a mean-reverting component in the underlying asset diffusion. Our work also extends the Fong and Vasicek (1992) model since we value options on general mean-reverting underlying assets semi-analytically.

However, adding a mean-reverting feature in our models only allows getting a semi-closed-form characteristic function, in the sense that we propose to solve some ODEs with numerical methods like Runge-Kutta formula or Adams-Bashforth-Moulton method. In our work, the numerical resolution is very accurate and takes much less time than the exact computation since analytic solutions (when they exist) involve complex algebra with Whittaker and the confluent hypergeometric functions. The use of numerical integration method, like the Gaussian-Laguerre quadrature rule, to invert the characteristic function is necessary to recover cumulative probabilities and then to price derivatives. With some pricing applications within different frameworks as Black and Scholes (1973), Longstaff and Schwartz (1995), Heston (1993) and Zhu (2000), it is shown that a good accuracy is achieved even with small quadrature rule orders and that the relative pricing errors are very small and convergent to 0. These results also apply to our mean-reverting frameworks, the prices converge to asymptotic prices and the relative errors are small and tend to 0.

As particular applications to our general valuation models, we derive semi-closed-form pricing formulas for credit-spread European options, caps, floors and swaps. We also show an interesting feature of derivative prices on mean-reverting assets. We find that the
impact of small reversion coefficients on the prices could be very large. This finding proves that the pricing of derivatives on mean-reverting underlying assets is very sensitive to the strength of the reversion and that has to be taken into account.

The combination of numerical resolution of ODEs with numerical integration using Gaussian-Laguerre quadrature rule provides extremely accurate valuation of credit derivatives and may do well for derivatives on other mean-reverting underlying assets like interest rates and commodities.
References


Table 1: Convergence to B&S ATM call

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Table 1 presents the results of the valuation of an at-the-money call within Black and Scholes (1973) framework both with the B&S analytic formula and with our numerical procedure. The option’s parameters are $X = \ln(100)$; $K = 100$; $r = 0.05$ and $\nu = 0.04$. To match B&S framework, the model’s parameters are $\mu = r$; $\alpha = 0$; $\gamma = 0.5$; $\rho = 0$; $\sigma = 0$; $\lambda = 0$; $\kappa = 0$ and $\theta = 0$. 
Table 2: Convergence to L&S ATM call

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Table 2 presents the results of the valuation of an at-the-money call within Longstaff and Schwartz (1995) framework both with the L&S analytic formula and with our numerical procedure. The option’s parameters are \( X = \ln(0.02); \ K = 0.02; \ r = 0.05 \) and \( V = 0.04 \). To match L&S framework, the model’s parameters are \( \mu = 0.02; \ \alpha = 0.015; \ \gamma = 0; \ \rho = 0; \ \sigma = 0; \ \lambda = 0; \ \kappa = 0 \) and \( \theta = 0 \).
Table 3 : Convergence to Zhu Square-Root ATM call

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Table 3 presents the results of the valuation of an at-the-money call within Zhu (2000) framework both with the Zhu square-root semi-analytic formula and with our numerical procedure. The option’s parameters are \( X = \ln(100) \); \( K = 100 \); \( r = 0.05 \) and \( V = 0.04 \). To match Zhu framework, the model’s parameters are \( \mu = r \); \( \alpha = 0 \); \( \gamma = 0.5 \); \( \rho = -0.5 \); \( \sigma = 0.1 \); \( \lambda = 4 \); \( \kappa = 4 \) and \( \theta = 0.06 \).
Table 4: Convergence to Zhu Ornstein-Uhlenbeck ATM call

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</table>

Table 4 presents the results of the valuation of an at-the-money call within Zhu (2000) framework both with the Zhu Ornstein-Uhlenbeck semi-analytic formula and with our numerical procedure. The option’s parameters are $X = \ln(100)$; $K = 100$; $r = 0.05$ and $\sigma = 0.2$. To match Zhu framework, the model’s parameters are $\mu = r$; $\alpha = 0$; $\gamma = 0.5$; $\rho = -0.5$; $\beta = 0.1$; $\lambda = 4$; $\kappa = 4$ and $\theta = 0.06$. 

33
Table 5: Price of an ATM call under the Square-Root Mean-Reverting model

<table>
<thead>
<tr>
<th>Maturity</th>
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<th>6 months</th>
<th>9 months</th>
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Table 5 presents the results of the valuation of an at-the-money call within our square-root mean-reverting framework. The option’s parameters are $X = \ln(0.02)$; $K = 0.02$; $r = 0.05$ and $V = 0.04$. The model’s parameters are $\mu = 0.03$; $\alpha = 0.02$; $\gamma = 0$; $\rho = -0.5$; $\sigma = 0.2$; $\lambda = 1$; $\kappa = 1$ and $\theta = 0.05$. 
Table 6: Price of an ATM call under the Ornstein-Uhlenbeck Mean-Reverting model

<table>
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<th>Maturity</th>
<th>3 months</th>
<th>6 months</th>
<th>9 months</th>
<th>1 year</th>
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</thead>
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<td>3.025470E-03</td>
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</table>

Table 6 presents the results of the valuation of an at-the-money call within our Ornstein-Uhlenbeck mean-reverting framework. The option’s parameters are $X = \ln(0.02)$; $K = 0.02$; $r = 0.05$ and $\sigma = 0.2$. The model’s parameters are $\mu = 0.03$; $\alpha = 0.02$; $\gamma = 0$; $\rho = -0.5$; $\beta = 0.2$; $\lambda = 1$; $\kappa = 1$ and $\theta = 0.05$. 
Table 7: Impact of the mean-reversion on the Call price under the Square-Root Mean-Reverting model

Table 7 presents the impact of the mean-reversion coefficient within our square-root mean-reverting framework. The relative difference is computed with respect to the “no mean-reversion price” i.e. $\alpha = 0$. The option’s parameters are $K = 0.02$; $r = 0.05$ and $V = 0.04$. The model’s parameters are $\mu = 0.03$; $\gamma = 0$; $\rho = -0.5$; $\sigma = 0.2$; $\lambda = 1$; $\kappa = 1$ and $\theta = 0.05$.

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<tr>
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<td>12%</td>
<td>20%</td>
<td>25%</td>
<td>30%</td>
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</tbody>
</table>
Table 8 : Impact of the mean-reversion on the Call price
under the Ornstein-Uhlenbeck Mean-Reverting model

Table 8 presents the impact of the mean-reversion coefficient within our Ornstein-
Uhlenbeck mean-reverting framework. The relative difference is computed with respect to
the “no mean-reversion price” i.e. $\alpha = 0$. The option’s parameters are $K = 0.02$; $r = 0.05$
and $V = 0.04$. The model’s parameters are $\mu = 0.03$; $\gamma = 0$; $\rho = -0.5$; $\beta = 0.2$; $\lambda = 1$;
$\kappa = 1$ and $\theta = 0.05$.

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<tr>
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Figures

Figure 1: B&S call pricing relative error

Figure 1 shows the relative pricing error of a call within Black and Scholes (B&S) framework. The true price is given by B&S analytic formula. The underlying asset values are \( X = \ln(100); \) \( X = \ln(110) \) and \( X = \ln(90) \). The option’s parameters are \( K = 100; \) \( T = 0.5; \) \( r = 0.05 \) and \( \nu = 0.04 \). To match B&S framework, the model’s parameters are \( \mu = r; \) \( \alpha = 0; \) \( \gamma = 0.5; \) \( \rho = 0; \) \( \sigma = 0; \) \( \lambda = 0; \) \( \kappa = 0 \) and \( \theta = 0 \).
Figure 2 shows the relative pricing error of a call within Longstaff and Schwartz (L&S) framework. The true price is given by L&S analytic formula. The underlying asset values are $X = \ln(0.02)$; $X = \ln(0.022)$ and $X = \ln(0.018)$. The option’s parameters are $K = 0.02$; $T = 0.5$; $r = 0.05$ and $V = 0.04$. To match L&S framework, the model’s parameters are $\mu = 0.02$; $\alpha = 0.015$; $\gamma = 0$; $\rho = 0$; $\sigma = 0$; $\lambda = 0$; $\kappa = 0$ and $\theta = 0$. 
Figure 3 shows the relative pricing error of a call within Zhu framework. The true price is given by Zhu square-root semi-analytic formula. The underlying asset values are $X = \ln(100)$; $X = \ln(120)$ and $X = \ln(80)$. The option’s parameters are $K = 100$; $T = 0.5$; $r = 0.05$ and $V = 0.04$. To match Zhu framework, the model’s parameters are $\mu = 0.05$; $\alpha = 0$; $\gamma = 0.5$; $\rho = -0.5$; $\sigma = 0.1$; $\lambda = 4$; $\kappa = 4$ and $\theta = 0.06$. 
Figure 4 shows the relative pricing error of a call within Zhu framework. The true price is given by Zhu Ornstein-Uhlenbeck semi-analytic formula. The underlying asset values are $X = \ln(100)$; $X = \ln(120)$ and $X = \ln(80)$. The option’s parameters are $K = 100$; $T = 1$; $r = 0.05$ and $\sigma = 0.2$. To match Zhu framework, the model’s parameters are $\mu = 0.05$; $\alpha = 0$; $\gamma = 0.5$; $\rho = -0.5$; $\beta = 0.1$; $\lambda = 4$; $\kappa = 4$ and $\theta = 0.06$. To keep the same scale, the out-of-the-money pricing relative error is divided by 50.
Figure 5 shows the relative pricing error of a call within our square-root mean-reverting framework. The true price is the asymptotic price. The underlying asset values are $X = \ln(0.02)$; $X = \ln(0.025)$ and $X = \ln(0.018)$. The option’s parameters are $K = 0.02$; $T = 0.5$; $r = 0.05$ and $V = 0.04$. The model’s parameters are $\mu = 0.03$; $\alpha = 0.02$; $\gamma = 0$; $\rho = -0.5$; $\sigma = 0.2$; $\lambda = 1$; $\kappa = 1$ and $\theta = 0.05$. 
Figure 6 shows the relative pricing error of a call within our Ornstein-Uhlenbeck mean-reverting framework. The true price is the asymptotic price. The underlying asset values are $X = \ln(0.02)$; $X = \ln(0.025)$ and $X = \ln(0.018)$. The option’s parameters are $K = 0.02$; $T = 1$; $r = 0.05$ and $\sigma = 0.2$. The model’s parameters are $\mu = 0.03$; $\alpha = 0.02$; $\gamma = 0$; $\rho = -0.5$; $\beta = 0.2$; $\lambda = 1$; $\kappa = 1$ and $\theta = 0.05$. 
Appendix A.1 : Derivation of the square-root mean-reverting characteristic function

The model is given under the risk-neutral measure $Q$ by:

$$dX_t = (\mu - \alpha X_t - \gamma V_t)dt + \sqrt{V_t}dW_1(t)$$

$$dV_t = (\kappa \theta - \lambda V_t)dt + \sigma \sqrt{V_t}dW_2(t)$$

where $d\langle W_1, W_2 \rangle_t = \rho dt$. If we define $Y_t = e^{\alpha t} X_t$, by Ito’s lemma we have:

$$dY_t = e^{\alpha t} (\mu - \gamma V_t)dt + e^{\alpha t} \sqrt{V_t}dW_1(t)$$

Solving the SDE gives:

$$Y_T = Y_i + \int_t^T e^{\alpha s} (\mu - \gamma V_s)ds + \int_t^T e^{\alpha s} \sqrt{V_s}dW_1(s)$$

$$Y_T = Y_i + \frac{\mu}{\alpha} (e^{\alpha T} - e^{\alpha i}) - \gamma \int_t^T e^{\alpha s} V_s ds + \int_t^T e^{\alpha s} \sqrt{V_s}dW_1(s)$$

The process $X$ is then expressed as:

$$X_T = e^{-\alpha(T-t)} X_i + \frac{\mu}{\alpha} (1 - e^{-\alpha(T-t)}) - \gamma \int_t^T e^{-\alpha(T-s)} V_s ds + \int_t^T e^{-\alpha(T-s)} \sqrt{V_s}dW_1(s)$$

So we have that:

$$\exp(\psi X_T) = \exp\left(\psi e^{-\alpha(T-t)} X_i + \frac{\mu}{\alpha} \psi (1 - e^{-\alpha(T-t)}) \right) \times \exp\left(-\gamma \psi \int_t^T e^{-\alpha(T-s)} V_s ds \right)$$

$$\times \exp\left(\psi \int_t^T e^{-\alpha(T-s)} \sqrt{V_s}dW_1(s) \right)$$

Since $W_1$ and $W_2$ are correlated, we can write:

$$dW_1(s) = \rho dW_2(s) + \sqrt{1-\rho^2}dW(s)$$

where $W$ and $W_2$ are uncorrelated Brownian motions. We then obtain:
\[ E_t^Q (e^{\psi X_t}) = \exp \left( \psi e^{-\alpha(T-t)} X_t + \frac{\mu}{\alpha} \psi (1 - e^{-\alpha(T-t)}) \right) \]

\[ \times E_t^Q \left[ \exp \left( -\gamma \psi \int_t^T e^{-\alpha(T-s)} V_s \, ds \right) \times \exp \left( \rho \psi \int_t^T e^{-\alpha(T-s)} \sqrt{V_s} \, dW_2(s) \right) \right] \]

\[ \times \exp \left( \psi \sqrt{1 - \rho^2} \int_t^T e^{-\alpha(T-s)} \sqrt{V_s} \, dW(s) \right) \]

\[ \times \mathcal{E} \left( W_2(s) \mid 0 \leq s \leq T \right) \]

Since \( W \) and \( W_2 \) are independent, \( (W(s))_{0 \leq s \leq T} \perp \mathcal{E} \left( W_2(s) \mid 0 \leq s \leq T \right) \), the equation becomes:

\[ E_t^Q (e^{\psi X_t}) = \exp \left( \psi e^{-\alpha(T-t)} X_t + \frac{\mu}{\alpha} \psi (1 - e^{-\alpha(T-t)}) \right) \]

\[ \times E_t^Q \left[ \exp \left( -\gamma \psi \int_t^T e^{-\alpha(T-s)} V_s \, ds \right) \times \exp \left( \rho \psi \int_t^T e^{-\alpha(T-s)} \sqrt{V_s} \, dW_2(s) \right) \right] \]

\[ \times E_t^Q \left[ \exp \left( \psi \sqrt{1 - \rho^2} \int_t^T e^{-\alpha(T-s)} \sqrt{V_s} \, dW(s) \right) \right] \]

\[ \times \mathcal{E} \left( W_2(s) \mid 0 \leq s \leq T \right) \]

At this stage, we did not need the particular square-root specification of the volatility diffusion. These equations will be also valid for the Ornstein-Uhlenbeck volatility diffusion.

For the square-root model, by Ito’s lemma we can write for the squared volatility:

\[ \text{At this stage, we did not need the particular square-root specification of the volatility diffusion. These equations will be also valid for the Ornstein-Uhlenbeck volatility diffusion.} \]
\[ d(e^{-\alpha(T-t)}V_t) = e^{-\alpha(T-t)}(\kappa \theta + (\alpha - \lambda)V_t)dt + e^{-\alpha(T-t)}\sigma \sqrt{V_t}dW_2(t) \]

Integrating this SDE and re-arranging it leads to:

\[
\sigma \int_t^T e^{-\alpha(T-s)} \sqrt{V_s} dW_2(s) = V_T - e^{-\alpha(T-t)}V_t - \int_t^T e^{-\alpha(T-s)}(\kappa \theta + (\alpha - \lambda)V_s)ds
\]

We then obtain:

\[
E_t^Q(e^{\psi X_T}) = \exp \left( \psi e^{-\alpha(T-t)}X_t + \frac{\mu}{\alpha} \psi \left( 1 - e^{-\alpha(T-t)} \right) \right) \times E_t^Q \left( \psi V_T - \frac{\rho}{\sigma} \psi \left( \alpha - \lambda \right) \int_t^T e^{-\alpha(T-s)}V_sds \right) \\
= \exp \left( \psi e^{-\alpha(T-t)}X_t + \frac{\mu}{\alpha} \psi \left( 1 - e^{-\alpha(T-t)} \right) - \frac{\rho}{\sigma} \psi e^{-\alpha(T-t)}V_t - \frac{\rho}{\sigma} \frac{\kappa \theta}{\alpha} \psi \left( 1 - e^{-\alpha(T-t)} \right) \right) \times E_t^Q \left( \frac{\rho}{\sigma} \psi V_T \right) \exp \left( \int_t^T \frac{1}{2} \psi^2 \left( 1 - \rho^2 \right) e^{-2\alpha(T-s)} - \left( \frac{\rho}{\sigma} \left( \alpha - \lambda \right) + \gamma \right) \psi e^{-\alpha(T-t)} \right] V_sds \right) \\

We can rewrite this equation as:

\[
E_t^Q(e^{\psi X_T}) = \exp \left( \psi e^{-\alpha(T-t)}X_t + \frac{\mu}{\alpha} \psi \left( 1 - e^{-\alpha(T-t)} \right) - \frac{\rho}{\sigma} \psi e^{-\alpha(T-t)}V_t - \frac{\rho}{\sigma} \frac{\kappa \theta}{\alpha} \psi \left( 1 - e^{-\alpha(T-t)} \right) \right) \times E_t^Q \left( \varepsilon_2 V_T - \int_t^T \varepsilon_1(T-s) V_s ds \right)
\]

where

\[
\begin{align*}
\varepsilon_1(\tau) &= \left( \frac{\rho}{\sigma} \left( \alpha - \lambda \right) + \gamma \right) \psi \exp(-\alpha \tau) - \frac{1}{2} \psi^2 \left( 1 - \rho^2 \right) \exp(-2\alpha \tau) \\
\varepsilon_2 &= \frac{\rho}{\sigma} \psi
\end{align*}
\]
Define the function $F(t,V)$ by:

$$F(t,V) = E_{t,V}^{0} \left[ \exp \{ \varepsilon_{2} V_{T} \} \exp \left\{ - \int_{t}^{T} \varepsilon_{1} (T-s)V_{s} \, ds \right\} \right]$$

then by Feynman-Kac theorem, we have that $F(t,V)$ must satisfy the following PDE:

$$\begin{cases}
\frac{\partial F}{\partial t} + \frac{1}{2} \sigma^2 V \frac{\partial^2 F}{\partial V^2} + (\kappa \theta - \lambda V) \frac{\partial F}{\partial V} - \varepsilon_{1} (T - t) V F = 0 \\
F(T,V) = \exp(\varepsilon_{2} V)
\end{cases}$$

Replacing the time variable $t$ by $\tau = T - t$, we can rewrite this PDE as (without ambiguity, we keep the same function’s $F$):

$$\begin{cases}
\frac{\partial F}{\partial \tau} = \frac{1}{2} \sigma^2 V \frac{\partial^2 F}{\partial V^2} + (\kappa \theta - \lambda V) \frac{\partial F}{\partial V} - \varepsilon_{1} (\tau) V F \\
F(0,V) = \exp(\varepsilon_{2} V)
\end{cases}$$

If we assume that $F(\tau,V)$ is log-linear and given by:

$$F(\tau,V) = \exp[D(\tau)V + C(\tau)]$$

where

$$D(0) = \varepsilon_{2} \quad ; \quad C(0) = 0$$

we have:

$$\begin{cases}
\frac{\partial F}{\partial \tau} = [D'(\tau)V + C'(\tau)] \times F(\tau,V) \\
\frac{\partial F}{\partial V} = D(\tau) \times F(\tau,V) \\
\frac{\partial^2 F}{\partial V^2} = D^2(\tau) \times F(\tau,V)
\end{cases}$$
The PDE for $F(\tau, V)$ becomes:

$$D'(\tau)V + C'(\tau) = \frac{1}{2} \sigma^2 D^2(\tau) + (\kappa \theta - \lambda V)D(\tau) - V \epsilon_1(\tau)$$

After re-arranging it as a polynomial of $V$, we deduce the ODEs satisfied by $D(\tau)$:

$$\begin{cases}
D'(\tau) - \frac{1}{2} \sigma^2 D^2(\tau) + \lambda D(\tau) + \epsilon_1(\tau) = 0 \\
D(0) = \epsilon_2 = \frac{\rho}{\sigma} \psi
\end{cases}$$

and by $C(\tau)$:

$$\begin{cases}
C'(\tau) - \kappa \theta D(\tau) = 0 \\
C(0) = 0
\end{cases}$$

The actualized characteristic function is then given by:

$$f(\psi) \equiv E^0\left[ \exp\left( -\int_{\tau_0}^{\tau} r(s)ds \right) \exp(\psi X_{\tau}) \right]$$

$$= e^{-r(T-t_0)} \times \exp\left( \psi e^{-\alpha(T-t_0)} X_t + \frac{\mu}{\alpha} \psi (1 - e^{-\alpha(T-t_0)}) - \frac{\rho}{\sigma} \psi e^{-\alpha(T-t_0)} V_t - \frac{\rho}{\sigma} \kappa \theta \psi (1 - e^{-\alpha(T-t_0)}) \right) \times \exp[D(T-t; \psi)V_t + C(T-t; \psi)]$$
Appendix A.2 : Exact resolution of the ODEs satisfied by \( D \) and \( C \) in the square-root framework

The ODEs satisfied by the functions \( D \) and \( C \) are:

\[
\begin{align*}
D'(\tau) - \frac{1}{2} \sigma^2 D^2(\tau) + \lambda D(\tau) + \epsilon_i(\tau) &= 0 \\
D(0) &= \frac{\rho}{\sigma} \psi
\end{align*}
\]

and

\[
\begin{align*}
C'(\tau) - \kappa \theta D(\tau) &= 0 \\
C(0) &= 0
\end{align*}
\]

Making the traditional (for Riccati-type ODEs) following transformation:

\[
U(\tau) = \exp\left(-\frac{\sigma^2}{2} \int D(s) ds\right)
\]

leads to the following linear homogeneous second-order ODE:

\[
U''(\tau) + \lambda U'(\tau) - \frac{1}{2} \sigma^2 U(\tau) \epsilon_i(\tau) = 0
\]

Under this transformation we recover the original functions \( D \) and \( C \) simply by:

\[
\begin{align*}
D(\tau) &= -\frac{2}{\sigma^2} \frac{U'(\tau)}{U(\tau)} \\
C(\tau) &= -\frac{2\kappa \theta}{\sigma^2} \ln(U(\tau))
\end{align*}
\]

A further substitution \( V(z) \equiv V(\exp(-\alpha \tau)) \equiv U(\tau) \) reduces the ODE to:

\[
\alpha^2 z^2 V''(z) + \alpha(\alpha - \lambda) z V'(\tau) - \frac{1}{2} \sigma^2 V(z) \epsilon_i(z) = 0
\]
where \( e_1(z) \equiv \varepsilon_1(\tau) \). We then only need to solve for the function \( V(z) \). Softwares like \textit{Maple}® give the solution to this ODE in terms of special functions known as the Whittaker functions. These functions are related to the well-known confluent hypergeometric function (see Abramowitz and Stegun, 1968). The solution \( U \) is given by:

\[
U(\tau) \equiv V(e^{-\alpha \tau}) = A \exp(-d\alpha \tau)M(a, b, c \psi e^{-\alpha \tau}) + B \exp(-d\alpha \tau)W(a, b, c \psi e^{-\alpha \tau})
\]

where \( M(.) \) and \( W(.) \) are respectively the Whittaker\textit{M} and the Whittaker\textit{W} functions and:

\[
\begin{align*}
a &= -\frac{\rho \alpha + \gamma \sigma - \rho \lambda}{2\alpha \sqrt{\rho^2 - 1}} ; \quad b = \frac{\lambda}{2\alpha} \\
c &= \frac{\sigma \sqrt{\rho^2 - 1}}{\alpha} ; \quad d = \frac{\lambda - \alpha}{2\alpha}
\end{align*}
\]

Constants \( A \) and \( B \) are determined by writing down that \( D(0) = \frac{\rho}{\sigma} \psi \) and \( C(0) = 0 \).
Appendix B.1: Derivation of the Ornstein-Uhlenbeck mean-reverting characteristic function

The model is given under the risk-neutral measure $Q$ by:

$$dX_t = \left(\mu - \alpha X_t - \gamma \sigma_t^2\right) dt + \sigma_t dW_t(t)$$

$$d\sigma_t = (\kappa \theta - \lambda \sigma_t) dt + \beta dW_2(t)$$

where $d\langle W_1, W_2 \rangle_t = \rho dt$.

From Appendix A.1, we can write the characteristic function as:

$$E_t^Q \left(e^{\psi X_T} \right) = \exp \left(\psi e^{-\alpha(T-t)} X_t + \frac{\mu}{\alpha} \psi \left(1 - e^{-\alpha(T-t)}\right)\right) \times E_t^Q \left\{ \exp \left( -\gamma \psi \int_t^T e^{-\alpha(T-s)} \sigma_s^2 ds + \frac{1}{2} \psi^2 \left(1 - \rho^2\right) \int_t^T e^{-2\alpha(T-s)} \sigma_s^2 ds\right) \right\} \times \exp \left( \rho \psi \int_t^T e^{-\alpha(T-s)} \sigma_s dW_2(s) \right)$$

For the Ornstein-Uhlenbeck model, we can solve for the volatility:

$$e^{-\alpha(T-t)} \sigma_s dW_2(s) = e^{-\alpha(T-t)} \sigma_s \frac{\sigma_t}{\beta} d\sigma_s - e^{-\alpha(T-t)} \frac{\kappa \theta}{\beta} \sigma_s ds + e^{-\alpha(T-t)} \frac{\lambda}{\beta} \sigma_s^2 ds$$

Integrating this SDE and re-arranging it leads to:

$$\int_t^T e^{-\alpha(T-s)} \sigma_s dW_2(s) = \frac{1}{\beta} \int_t^T e^{-\alpha(T-s)} \sigma_s d\sigma_s - \frac{\kappa \theta}{\beta} \int_t^T e^{-\alpha(T-s)} \sigma_s ds + \frac{\lambda}{\beta} \int_t^T e^{-\alpha(T-s)} \sigma_s^2 ds$$

By Ito’s lemma, we also have:

$$d \left( e^{-\alpha(T-s)} \sigma_s^2 \right) = \alpha e^{-\alpha(T-s)} \sigma_s^2 ds + 2 e^{-\alpha(T-s)} \sigma_s d\sigma_s + \beta^2 e^{-\alpha(T-s)} ds$$

$$\frac{1}{\beta} \int_t^T e^{-\alpha(T-s)} \sigma_s d\sigma_s = \frac{1}{2\beta} \left[ \sigma_t^2 - e^{-\alpha(T-t)} \sigma_t^2 \right] - \frac{\alpha}{2\beta} \int_t^T e^{-\alpha(T-s)} \sigma_s^2 ds - \frac{\beta}{2\alpha} \left(1 - e^{-\alpha(T-t)}\right)$$

and
\[
\rho \psi^T e^{-\alpha(T-s)} \sigma_i dW_i(s) = \rho \psi \left( \frac{1}{2 \beta} \left[ \sigma_i^2 - e^{-\alpha(T-t)} \sigma_i^2 \right] - \frac{\alpha}{2 \beta} \int_t^T e^{-\alpha(T-s)} \sigma_i^2 ds - \frac{\beta}{2 \alpha} \left( 1 - e^{-\alpha(T-t)} \right) \right) \\
\quad - \frac{\rho \kappa \theta}{\beta} \psi \int_t^T e^{-\alpha(T-s)} \sigma_i ds + \rho \psi \frac{\lambda}{\beta} \int_t^T e^{-\alpha(T-s)} \sigma_i^2 ds \\
= \frac{\rho}{2 \beta} \psi \left[ \sigma_i^2 - e^{-\alpha(T-t)} \sigma_i^2 \right] + \frac{\rho}{\beta} \psi \left( \lambda - \frac{\alpha}{2} \right) \int_t^T e^{-\alpha(T-s)} \sigma_i^2 ds \\
\quad - \frac{\rho \beta}{2 \alpha} \psi \left( 1 - e^{-\alpha(T-t)} \right) - \frac{\rho \kappa \theta}{\beta} \psi \int_t^T e^{-\alpha(T-s)} \sigma_i ds \\
\]

The characteristic function is then given by:

\[
E_t^Q \left( e^{\psi X} \right) = \exp \left( \psi e^{-\alpha(T-t)} X_i + \frac{\mu}{\alpha} \psi \left( 1 - e^{-\alpha(T-t)} \right) - \frac{\rho}{2 \beta} \psi e^{-\alpha(T-t)} \sigma_i^2 - \frac{\rho \beta}{2 \alpha} \psi \left( 1 - e^{-\alpha(T-t)} \right) \right) \\
\quad \times E_t^Q \left( \exp \left( \frac{\rho}{2 \beta} \psi \sigma_i^2 - \frac{\rho \kappa \theta}{\beta} \psi \int_t^T e^{-\alpha(T-s)} \sigma_i ds \right) \right) \\
\quad \times \exp \left( \int_t^T \left[ \frac{1}{2} \psi^2 \left( 1 - \rho^2 \right) e^{-\alpha(T-s)} - \frac{\rho}{\beta} \left( \frac{\alpha}{2} - \lambda \right) + \gamma \right] e^{-\alpha(T-s)} \sigma_i^2 ds \right)
\]

We can rewrite this equation as:

\[
E_t^Q \left( e^{\psi X} \right) = \exp \left( \psi e^{-\alpha(T-t)} X_i + \frac{\mu}{\alpha} \psi \left( 1 - e^{-\alpha(T-t)} \right) - \frac{\rho}{2 \beta} \psi e^{-\alpha(T-t)} \sigma_i^2 - \frac{\rho \beta}{2 \alpha} \psi \left( 1 - e^{-\alpha(T-t)} \right) \right) \\
\quad \times E_t^Q \left( \exp \left( \eta_3 \sigma_i^2 - \int_t^T \eta_2 (T-s) \sigma_i ds - \int_t^T \eta_1 (T-s) \sigma_i^2 ds \right) \right)
\]

where

\[
\eta_1(\tau) = \left( \frac{\alpha \rho}{2 \beta} - \frac{\rho \lambda}{\beta} + \gamma \right) \psi \exp(-\alpha \tau) - \frac{1}{2} \psi^2 \left( 1 - \rho^2 \right) \exp(-2 \alpha \tau) \\
\eta_2(\tau) = \frac{\rho \kappa \theta}{\beta} \psi \exp(-\alpha \tau) \\
\eta_3 = \frac{\rho}{2 \beta} \psi
\]
Define the function $G(t, \sigma)$ as :

$$G(t, \sigma) = E_t^G \left( \exp \left( \eta_3 \sigma^2 - \int_t^T \eta_2 (T-s) \sigma_s ds - \int_t^T \eta_1 (T-s) \sigma_s^2 ds \right) \right)$$

By Feynman-Kac theorem, we have that $G(t, \nu)$ must satisfy the following PDE :

$$\begin{cases}
\frac{\partial G}{\partial t} + \frac{1}{2} \beta^2 \frac{\partial^2 G}{\partial \sigma^2} + (\kappa \theta - \lambda \sigma) \frac{\partial G}{\partial \sigma} - (\eta_1 (T-t) \sigma^2 + \eta_2 (T-t) \sigma) G = 0 \\
G(T, \sigma) = \exp(\eta_3 \sigma^2)
\end{cases}$$

Replacing the time variable $t$ by $\tau = T - t$ as before, we can rewrite this PDE as (without ambiguity, we keep the same function’s $G$) :

$$\begin{cases}
\frac{\partial G}{\partial \tau} = \frac{1}{2} \beta^2 \frac{\partial^2 G}{\partial \sigma^2} + (\kappa \theta - \lambda \sigma) \frac{\partial G}{\partial \sigma} - (\eta_1 (\tau) \sigma^2 + \eta_2 (\tau) \sigma) G \\
G(0, \sigma) = \exp(\eta_3 \sigma^2)
\end{cases}$$

Assuming that $G(\tau, \sigma)$ is log-linear and given by :

$$G(\tau, \sigma) = \exp \left[ \frac{1}{2} E(\tau) \sigma^2 + D(\tau) \sigma + C(\tau) \right]$$

where

$$E(0) = 2\eta_3 \quad ; \quad D(0) = 0 \quad ; \quad C(0) = 0$$

leads to :

$$\begin{cases}
\frac{\partial G}{\partial \tau} = \left[ \frac{1}{2} E'(\tau) \sigma^2 + D'(\tau) \sigma + C'(\tau) \right] \times G(\tau, \sigma) \\
\frac{\partial G}{\partial \sigma} = [E(\tau) \sigma + D(\tau)] \times G(\tau, \sigma) \\
\frac{\partial^2 G}{\partial \sigma^2} = E(\tau) \times G(\tau, \sigma) + [E(\tau) \sigma + D(\tau)]^2 \times G(\tau, \sigma)
\end{cases}$$
The PDE satisfied by $G(t, \sigma)$ becomes:

$$\frac{1}{2} E'(t) \sigma^2 + D'(t) \sigma + C'(t) = \frac{1}{2} \beta^2 [E(t) + (E(t) \sigma + D(t))^2]$$

$$+ (\kappa \theta - \lambda \sigma)[E(t) \sigma + D(t)]$$

$$- [\eta_2(t) \sigma + \eta_1(t) \sigma^2]$$

After re-arranging it as a polynomial of $\sigma$, we deduce the ODEs satisfied by $E(t)$:

$$\begin{cases}
\frac{1}{2} E'(t) - \frac{1}{2} \beta^2 E^2(t) + \lambda E(t) + \eta_1(t) = 0 \\
E(0) = 2\eta_3 = \frac{\rho}{\beta} \psi
\end{cases}$$

by $D(t)$:

$$\begin{cases}
D'(t) - \beta^2 E(t)D(t) + \lambda D(t) - \kappa \theta E(t) + \eta_2(t) = 0 \\
D(0) = 0
\end{cases}$$

and by $C(t)$:

$$\begin{cases}
C'(t) - \frac{1}{2} \beta^2 E(t) - \frac{1}{2} \beta^2 D^2(t) - \kappa \theta D(t) = 0 \\
C(0) = 0
\end{cases}$$

The actualized characteristic function is then given by:

$$f(\psi) = E_t^Q \left[ \exp \left( - \int_t^T r(s) ds \right) \exp(\psi X_T) \right]$$

$$= e^{-r(T-t)} \exp \left( \psi \mu \alpha \left( 1 - e^{-\alpha(t-T)} \right) - \frac{\rho}{2\beta} \psi^2 \left( 1 - e^{-\alpha(t-T)} \right) \sigma_t^2 \right)$$

$$\times \exp \left[ \frac{1}{2} E(T-t; \psi) \sigma_t^2 + D(T-t; \psi) \sigma_t + C(T-t; \psi) \right]$$
Appendix B.2 : Exact resolution of the ODE satisfied by \( E \) in the Ornstein-Uhlenbeck framework

The ODEs satisfied by the functions \( E \) is:

\[
\begin{align*}
\frac{1}{2} E'(\tau) - \frac{1}{2} \beta^2 E^2(\tau) + \lambda E(\tau) + \eta_1(\tau) &= 0 \\
E(0) &= 2\eta_1 = \frac{\rho}{\beta} \psi
\end{align*}
\]

As detailed in Appendix A.2, we use two transformations before getting the exact solution \( E \). Making the first transformation:

\[
U(\tau) = \exp\left(-\beta^2 \int E(s) ds \right)
\]

and the further substitution \( V(z) \equiv V(\exp(-\alpha \tau)) \equiv U(\tau) \) lead to:

\[
U(\tau) \equiv V(e^{-\alpha \tau}) = A \exp(-d\alpha \tau) M(a, b, c \psi e^{-\alpha \tau}) + B \exp(-d\alpha \tau) W(a, b, c \psi e^{-\alpha \tau})
\]

where again \( M(.) \) and \( W(.) \) are respectively the Whittaker\( M \) and the Whittaker\( W \) functions and:

\[
\begin{align*}
a &= -\frac{\rho \alpha + 2\gamma \beta - 2\rho \lambda}{2\alpha \sqrt{\rho^2 - 1}} \\
b &= \frac{\lambda}{\alpha} \\
c &= \frac{2\beta \sqrt{\rho^2 - 1}}{\alpha} \\
d &= \frac{2\lambda - \alpha}{2\alpha}
\end{align*}
\]

Constants \( A \) and \( B \) are determined by writing down that \( U(0) = 1 \) and \( U'(0) = -\beta \rho \psi \). We recover the function \( E \) by:

\[
E(\tau) = -\frac{1}{\beta^2} \frac{U'(\tau)}{U(\tau)}
\]
To see that neither $D$ (nor $C$) could be expressed in a closed-form way, we use a traditional technique to solve linear first-degree ODEs to find:

$$D(\tau) = \frac{\kappa \theta}{\beta^2} \frac{1}{e^{\lambda \tau} U(\tau)} \left( 1 - e^{\lambda \tau} U(\tau) + \int_0^{\varepsilon} (\lambda - \rho \beta \psi e^{-\alpha s}) e^{\lambda s} U(s) ds \right)$$

This expression could not be simplified further.
Appendix C: Proof of the well-definiteness of the integrands

For the purpose of integration, we must proof the well-definiteness of the function over the interval of integration, especially around the potential singularities. In particular, we have to value these two integrals:

\[
\int_0^{+\infty} \text{Re} \left( \frac{f(1+i\phi)}{i\phi f(1)} K^{-i\phi} \right) d\phi
\]

and

\[
\int_0^{+\infty} \text{Re} \left( \frac{f(i\phi)}{i\phi f(0)} K^{-i\phi} \right) d\phi
\]

where \( f \) is a characteristic function, which is of class \( C^\infty([0,+\infty[) \), defined by:

\[
f(\psi) = E^Q_t \left\{ \exp \left\{ -\int_{t}^{T} r(s) ds \right\} \exp\{\psi X_T\} \right\}
\]

Using a Taylor expansion around \( \phi = 0 \), the two integrands tend respectively to:

\[
\text{Re} \left( \frac{f(i\phi)}{i\phi f(0)} K^{-i\phi} \right) \rightarrow \frac{f'(0)}{f(0)} - \ln(K) = \frac{E^Q_t \left\{ X_T e^{-\int_{t}^{T} r(s) ds} \right\}}{E^Q_t \left\{ e^{-\int_{t}^{T} r(s) ds} \right\}} - \ln(K)
\]

and

\[
\text{Re} \left( \frac{f(1+i\phi)}{i\phi f(1)} K^{-i\phi} \right) \rightarrow \frac{f'(1)}{f(1)} - \ln(K) = \frac{E^Q_t \left\{ e^{-\int_{t}^{T} r(s) ds} X_T \exp\{X_T\} \right\}}{E^Q_t \left\{ e^{-\int_{t}^{T} r(s) ds} \exp\{X_T\} \right\}} - \ln(K)
\]
Appendix D : Feynman-Kac theorem (Karatzas and Shreve 1991)

Under some regularity assumptions, if we suppose that $F(t, V):[0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$
is of class $C^{1,2}([0, T] \times \mathbb{R}^d)$ and satisfies the Cauchy problem :

$$\begin{cases}
\frac{\partial F}{\partial t} + A_t F + g(t, V) - h(t, V) F(t, V) = 0 \\
F(t, V) = k(V)
\end{cases}$$

where $A_t$ is the second order differential operator, then $F(t, V)$ is unique and admits the stochastic representation :

$$F(t, V) = E_{t,V} \left[ k(V_T) \exp \left\{ - \int_t^T h(s, V_s) ds \right\} \right]$$

$$+ \int_t^T g(\tau, V_\tau) \exp \left\{ - \int_t^\tau h(s, V_s) ds \right\} d\tau$$
Appendix E: Derivation of the Greeks

Recall that the Call premium is given by:

\[ \text{Call}(t, T) = f(t, T; 1) Q^T_1(X_T > \ln(K)) - f(t, T; 0) K Q^T_2(X_T > \ln(K)) \]

For simplicity, we denote \( f(t, T; \psi) \) by \( f(\psi) \) and \( Q^T_j(X_T > \ln(K)) \) by \( Q_j \). We need to compute the following derivatives:

\[
\frac{\partial f(\psi)}{\partial X} = \psi e^{-\alpha(T-t)} f(\psi)
\]

\[
\frac{\partial \left( \frac{f(\psi)}{f(1)} \right)}{\partial X_t} = (\psi - 1)e^{-\alpha(T-t)} \frac{f(\psi)}{f(1)}
\]

\[
\frac{\partial Q_1}{\partial X_t} = \frac{e^{-\alpha(T-t)}}{\pi} \int_0^{+\infty} \text{Re} \left( \frac{f(1 + i\phi)}{f(1)} K^{-i\phi} \right) d\phi
\]

\[
\frac{\partial Q_2}{\partial X_t} = \frac{e^{-\alpha(T-t)}}{\pi} \int_0^{+\infty} \text{Re} \left( \frac{f(i\phi)}{f(0)} K^{-i\phi} \right) d\phi
\]

We then can write:

\[
\text{Delta}(t, T) \equiv \frac{\partial \text{Call}(t, T)}{\partial e^X_t}
\]

\[
= e^{-X} \frac{\partial \text{Call}(t, T)}{\partial X_t}
\]

\[
= e^{-X} \left[ \frac{\partial f(1)}{\partial X_t} Q_1 + f(1) \frac{\partial Q_1}{\partial X_t} - f(0) K \frac{\partial Q_2}{\partial X_t} \right]
\]

By a "formal" change of variable, we can show that:

\[
f(1) \frac{\partial Q_1}{\partial X_t} = f(0) K \frac{\partial Q_2}{\partial X_t}
\]

and we can deduce the formula for the Delta as given in the main text.
Define the Gamma by:

\[ \Gamma(t,T) \equiv \frac{\partial \Delta(t,T)}{\partial e^{X_t}} \]

Using the same calculus done to derive the Delta leads to:

\[ \Gamma(t,T) = e^{-X_t} \frac{\partial \Delta(t,T)}{\partial X_t} \]

\[ = e^{-X_t} \begin{bmatrix} -e^{-\alpha(T-t)} e^{-X_t} f(1) Q_1 + e^{-\alpha(T-t)} e^{-X_t} \frac{\partial f(1)}{\partial X_t} Q_1 \\ + e^{-\alpha(T-t)} e^{-X_t} f(1) \frac{\partial Q_1}{\partial X_t} \end{bmatrix} \]

\[ = \left( 1 - e^{-\alpha(T-t)} \right) e^{-X_t} \Delta(t,T) + \frac{e^{-2\alpha(T-t)} e^{-2X_t}}{\pi} \int_0^{+\infty} \text{Re} \left( f(1+i\phi) K^{-i\phi} \right) d\phi \]

For the Vega, consider first the square-root model:

\[ \nu(t,T) \equiv \frac{\partial \text{Call}(t,T)}{\partial V_t} \]

We compute the following derivatives:

\[ \frac{\partial f(\psi)}{\partial V_t} = \left( D(T-t;\psi) - \frac{\rho}{\sigma} \psi e^{-\alpha(T-t)} \right) f(\psi) \]

\[ \frac{\partial}{\partial V_t} \left( \frac{f(\psi)}{f(1)} \right) = \left( D(T-t;\psi) - D(T-t;1) - \frac{\rho}{\sigma} (\psi - 1) e^{-\alpha(T-t)} \right) \frac{f(\psi)}{f(1)} \]

\[ \frac{\partial Q_1}{\partial V_t} = \frac{1}{\pi} \int_0^{+\infty} \text{Re} \left[ \left( \frac{D(T-t;1+i\phi) - D(T-t;1)}{i\phi} - \frac{\rho}{\sigma} e^{-\alpha(T-t)} \right) \frac{f(1+i\phi)}{f(1)} K^{-i\phi} \right] d\phi \]

\[ \frac{\partial Q_2}{\partial V_t} = \frac{1}{\pi} \int_0^{+\infty} \text{Re} \left[ \left( \frac{D(T-t;i\phi) - \frac{\rho}{\sigma} e^{-\alpha(T-t)}}{i\phi} \right) \frac{f(i\phi)}{f(0)} K^{-i\phi} \right] d\phi \]
We now can write for the Vega :

\[
Vega(t, T) = \frac{\partial f(1)}{\partial V_t} Q_1 + \frac{f(1)}{\partial V_t} \frac{\partial Q_1}{\partial V_t} - f(0) K \frac{\partial Q_2}{\partial V_t} \\
= \left( D(T - t; 1) - \frac{\rho}{\sigma} e^{-\alpha(T-t)} \right) f(1) Q_1 \\
+ \frac{1}{\pi} \int_0^{+\infty} \text{Re} \left[ \frac{D(T-t; 1 + i\phi) - D(T-t; 1)}{i\phi} f(1 + i\phi) K^{-\phi} \right] d\phi \\
- \frac{K}{\pi} \int_0^{+\infty} \text{Re} \left[ \frac{D(T-t; i\phi)}{i\phi} f(i\phi) K^{-\phi} \right] d\phi
\]

For the Ornstein-Uhlenbeck model, define the Vega by :

\[
Vega(t, T) \equiv \frac{\partial \text{Call}(t, T)}{\partial \sigma_t}
\]

The following derivatives are needed :

\[
\frac{\partial f(\psi)}{\partial \sigma_t} = \left( E(T-t; \psi) \sigma_t + D(T-t; \psi) - \frac{\rho}{\beta} \psi e^{-\alpha(T-t)} \right) f(\psi)
\]

\[
\frac{\partial \left( \frac{f(\psi)}{f(1)} \right)}{\partial \sigma_t} = \left( \frac{\left[ E(T-t; \psi) - E(T-t; 1) \right] \sigma_t + D(T-t; \psi) - D(T-t; 1)}{-\frac{\rho}{\beta} (\psi - 1) e^{-\alpha(T-t)} \sigma_t} \right) \frac{f(\psi)}{f(1)}
\]

\[
\frac{\partial Q_1}{\partial V_t} = \frac{1}{\pi} \int_0^{+\infty} \text{Re} \left[ \frac{E(T-t; 1 + i\phi) - E(T-t; 1)}{i\phi} \sigma_t \right. \\
\left. + \frac{D(T-t; 1 + i\phi) - D(T-t; 1)}{i\phi} - \frac{\rho}{\beta} e^{-\alpha(T-t)} \right] \frac{f(1 + i\phi) K^{-\phi}}{f(1)} d\phi
\]

\[
\frac{\partial Q_2}{\partial V_t} = \frac{1}{\pi} \int_0^{+\infty} \text{Re} \left[ \frac{E(T-t; i\phi)}{i\phi} \sigma_t + \frac{D(T-t; i\phi)}{i\phi} - \frac{\rho}{\sigma} e^{-\alpha(T-t)} \right] \frac{f(i\phi) K^{-i\phi}}{f(0)} d\phi
\]
The Vega can be written as:

\[
Vega(t,T) = \frac{\partial f(1)}{\partial \sigma} Q_1 + f(0) \frac{\partial Q_1}{\partial \sigma} - f(0) K \frac{\partial Q_2}{\partial \sigma},
\]

\[
= \left( E(T-t;1)\sigma + D(T-t;1) - \frac{\rho}{\beta} e^{-\alpha(T-t)} \sigma \right) f(1) Q_1
\]

\[
+ \frac{1}{\pi} \int_0^{+\infty} \text{Re} \left\{ \left[ \frac{E(T-t,1+i\phi)}{i\phi} - E(T-t,1) / \sigma \right] + \frac{D(T-t,1+i\phi) - D(T-t,1)}{i\phi} \right\} f(1+i\phi) K^{-i\phi} \, d\phi
\]

\[- \frac{K}{\pi} \int_0^{+\infty} \text{Re} \left\{ \left[ \frac{E(T-t;i\phi)}{i\phi} - \frac{D(T-t;i\phi)}{i\phi} \right] f(i\phi) K^{-i\phi} \right\} d\phi
\]

62
Footnotes

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