Valuing Defaultable Securities under Interest Rate and Default Risk Correlation

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Abstract
This paper studies the valuation of defaultable, callable bonds and credit default swaps when both interest rates and default intensity are stochastic. The model I adopt in the paper follows the framework of Duffie and Singleton (1999) and I determine the prices of these two defaultable securities numerically. Most work in the literature assumes zero correlation between the market and the credit risk. In my paper I allow for non-zero correlation between the two risks and examine the effect of this correlation on valuation and term structures of callable bonds and on default spreads. In addition, for defaultable, callable bonds, I examine the effects of different assumptions regarding recovery rate and the notice period on the valuation of callable bonds.
1 Introduction

There are two basic approaches to model the dynamic behavior of default risk. One approach -- so called structural models -- pioneered by Black and Scholes (1973) and Merton (1974) and extended by Black and Cox (1976), Shimko, Tejima, and Deventer (1993), Kim, Ramaswamy, and Sundaresan (1993), Longstaff and Schwartz (1995), Briys and De Varenne (1997), and others, explicitly models the evolution of firm value which is assumed to be observed by investors. Default is triggered when the value of the firm’s assets falls below or hits a pre-specified boundary. Although these models have proven very useful in examining credit risk, this class of models has been criticized for several reasons. First, since in most models firm value is described as a continuous diffusion process, default time is predictable. Therefore, credit spreads tend to be zero for the short-term debt of a solvent firm. This feature obviously contradicts empirical observations where credit spreads for short-term maturities of highly-rated firms remain strictly positive. Second, the issuer’s assets and liabilities are typically not traded in financial markets. So that their value is not directly observable and estimation of the firm value becomes problematic.

The second approach – also called the reduced-form approach -- adopted by Jarrow and Turnbull (1995), Duffie and Singleton (1997), Jarrow, Lando, and Turnbull (1997), Duffie (1998), Duffee (1999), Duffie and Singleton (1999), Madan and Unal (2000), and others, differs from the structural approach in the sense that the relation between default and firm value is not considered in a structural or explicit way. Instead, the default time is directly modeled as an unpredictable Poisson event. In particular, whenever the Poisson event occurs, the firm experiences a sudden loss in market value which could precipitate bankruptcy. The reduced-form approach is viewed as a viable alternative to the structural approach for several reasons. First, it is more tractable and easy to implement than the latter. Specifically, given an arbitrage-free setting and certain assumptions, defaultable bonds can be priced in the same way as default-free bonds. Second, the reduced-form approach does not require specifying the structure of the firm’s liabilities.
One of the major criticisms of the reduced-form approach is that since the default time is modeled as exogenous unpredictable Poisson event, this modeling approach has difficulty in answering questions such as: what cause firms to default? Duffie and Lando (2001) show that under the assumption of imperfect accounting information, which is more realistic than the assumption of perfect information, the structural and the reduced-form models show amazing similarity. Specifically, one may formally view the structural model with imperfect information as equivalent to the reduced-form model. This view helps remove the major objection to reduced-form models in the literature.

This paper follows the approach proposed by Duffie and Singleton (1999). Such a model, being similar to those commonly used to price default-free bonds and derivatives, allows for easy evaluation of defaultable claims. My objective in this paper is to apply the Duffie and Singleton model to price defaultable, callable bonds and credit default swaps. In addition, I explore the effect of market and credit risk correlation on the valuation of these two claims.

The rest of the paper is organized as follows. Section 2 describes the model structure. Section 3 specifies the model to be applied in the paper. Section 4 presents the valuation results of defaultable, callable bonds. Section 5 presents the pricing results credit default swaps. Section 6 summarizes the paper.

2 The model structure

The reduced-form credit risk model in Duffie and Singleton (1999) is the basis for valuing defaultable securities and derivatives in my paper. Trading can take place any time during the interval $[0, \bar{T}]$. Traded are default-free bonds and defaultable bonds of all maturities. Markets are assumed to be complete and frictionless, with no arbitrage opportunities. Under the assumptions of no arbitrage and complete markets, there exists a unique equivalent martingale measure $Q$ under which the market value of each security is the expectation of the discounted present value of its cash flows, using the compounded default-free short rate for discounting. For example, the value of a zero-coupon default-free bond, issued at date $t$ and maturing at date $T$, with promised payoff of 1 at maturity is
\[
\delta(t,T) = E_t^Q \{ e^{-\int_t^T r_{du}} \},
\]

where \( E_t^Q \) denotes risk-neutral expectation conditional on information known at date \( t \).

On the other hand, if the issuer defaults prior to the maturity date \( T \), then both the magnitude and timing of the payoff to investors may be uncertain. Let \( \tau \) denote the first time that this firm defaults, and let \( 1_{\{\tau > t\}} \) be the indicator of the event that \( \tau > t \), which takes the value of 1 if the issuer has not defaulted prior to time \( t \), and 0 otherwise. Then the value of this risky zero-coupon bond, with the promised payoff of 1 at maturity is

\[
V_0(t,T) = E_t^Q \{ e^{-\int_t^T r_{du}} 1_{\{\tau > t\}} + e^{-\int_t^\tau r_{du}} W 1_{\{\tau \leq T\}} \},
\]

where \( W \) is the value of recovery. The value of this risky debt is composed of two parts. The first part is the present value of the promised payment if default does not occur. The second part is the present value of the promised payment in default.

Depending on how the default time \( \tau \) is modeled, and how the recovery amount is specified, equation (2) could lead to different valuation formulas. This paper follows the model in Duffie (1998) and Duffie and Singleton (1999). First, I adopt the reduced-form approach, in which the timing of default is modeled as an unpredictable Poisson process with stochastic intensity, \( h(t) \), and \( h(t)\Delta \) approximates the probability of default over the next time period of length \( \Delta \), given that the firm has not defaulted yet at time \( t \). Then the conditional probability at time \( s \), given all available information at that time, of survival to time \( t \), is given by

\footnote{How to define the time of default \( \tau \) depends on which model is being used. The time of default \( \tau \) in structural-form models is the first hitting time of a diffusion process at a fixed barrier. The time of default \( \tau \) in reduced-form models is the time of the first jump of a Poisson process.}
\[ p(t \mid s) = E_s^Q [1_{\{T > t\}}] = E_s^Q \left[ e^{-\int_{u}^{t} \Phi(s) \, ds} \right]. \] (3)

Second, given that the commonly used reduced-form models differ mainly in their treatment of the recovery, I will focus on two assumptions on recovery: recovery of face value (RFV) and recovery of market value (RMV).

**Recovery of face value (RFV)**

The RFV assumption, which is studied in Duffie (1998), assumes that the recovery amount is a fraction of the face value of the claim. The value of recovery is expressed as

\[ W_t = (1 - L_t) \cdot 1, \] (4)

where \( L_t \) is the fractional loss in face value at time \( t \), and the value of risky zero-coupon bond is given by

\[ V_0^{RFV}(t, T) = E_t^Q \left[ e^{-\int_{t}^{T} 1 \cdot (1 - L_u) e^{-\int_{u}^{T} \Phi(s) \, ds} h_u \, du} \right] = E_t^Q \left[ e^{-\Phi(T)} + \int_{t}^{T} (1 - L_u) e^{-\Phi(u)} h_u \, du \right], \] (5)

where \( \Phi = r + h \) and \( \Phi(\tau) = \int_{t}^{\tau} \phi_s \, ds \). Furthermore, by adding deterministic and continuous coupon rate \( c(t) \) in my model, the value of this risky bond is

\[ V^{RFV}(t, T) = V_0^{RFV}(t, T) + E_t^Q \left[ \int_{t}^{T} c(u) e^{-\int_{u}^{T} \phi_s \, ds} \, du \right] = E_t^Q \left\{ e^{-\Phi(T)} + \int_{t}^{T} (1 - L_u) e^{-\Phi(u)} h_u \, du + \int_{t}^{T} c(u) e^{-\Phi(u)} \, du \right\}. \]

\( ^2 \) In addition to RFV and RMV, there is a third specification in modeling the recovery rate: recovery of treasury (RT). This approach assumes that, when default occurs, the debtholders recover a fraction of the value of an otherwise equivalent, default-free bond. See Jarrow and Turnbull (1995) for an example.

\( ^3 \) Please refer to Duffie and Singleton (1999) for details.
Recovery of market value (RMV)

The RMV assumption assumes that the recovery amount is a fraction of the market value of the same bond right before the default. Thus the value of recovery, \( W \), is given by

\[
W_t = (1 - L_t) v(t_-, T),
\]

where \( L_t \) denotes the expected fractional loss in market value at time \( t \), \( t_- \) represents an instant before default. Combining equation (2), (3) and (7) and under certain technical conditions, I obtain

\[
V_0^{RMV}(t, T) = E^Q_t [\exp(-\int_t^T R_s ds)] = E^Q_t [e^{-\Psi(T)}],
\]

where \( R_t = r_t + h_t L_t \) and \( \Psi(t) = \int_t^T R_u du \). Equation (8) implies that a defaultable zero-coupon bond may be priced as if it were default-free by replacing the usual short-term interest rate process \( r \) with the default-adjusted short-rate process \( R \). Discounting at this default-adjusted short-rate process \( R \) therefore accounts for the probability of default, timing of default, and the effect of loss on default. In the case of deterministic and continuous coupon rate \( c(t) \), the value of defaultable debt is given as

\[
V^{RMV}(t, T) = V_0^{RMV}(t, T) + \int_t^T c(u) V_0^{RMV}(t, u) du
\]

\[
= E^Q_t \left\{ e^{-\Psi(T)} + \int_t^T e^{-\Psi(u)} c(u) du \right\}.
\]

One may wonder about the implications of choosing one recovery assumption over the other. The assumption of RMV is easier to implement, because prices of defaultable

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4 For derivation details, please see Duffie and Singleton (1999).
bonds can be computed in a RMV model by the same formulas used for default-free bonds, using the default-adjusted short-rate process $R$ instead of the usual short-term interest rate process $r$. If, however, one assumes liquidation at default and that absolute priority applies, then the assumption of RFV is more realistic, as it assumes the same recovery rate for bonds of equal seniority by the same issuer. Duffie and Singleton (1999) and Skinner and Diaz (2000) point out that for the estimation of par bond spreads, or for the estimation of risk-neutral default intensities from par spreads, these two assumptions make little difference. In this paper, I examine whether these two assumptions make any difference when applied to the valuation of callable bonds.

3 My model specification
In this section, I describe a specific model which is used to value defaultable, callable bonds and credit default swaps in my paper later. Notice that this model allows for both default risk and interest rate risk and the correlation between these two risks is not necessarily zero.

**Assumption 1:** The interest rate $r_t$ follows a continuous, adapted Cox-Ingersoll-Ross (CIR, 1985) process

$$dr_t = \kappa_r (\theta_r - r_t)dt + \sigma_r \sqrt{r_t} dZ_{tr},$$  \hspace{1cm} (10)

where $Z_{tr}$ is a one-dimensional Brownian motion under the P-measure, $\kappa_r, \theta_r > 0$, and $2\kappa_r \theta_r \geq \sigma_r^2$. For pricing purpose I assume a constant market price of risk $\lambda$, which transforms equation (10) by the means of the Girsanov theorem into

$$dr_t = [\kappa_r (\theta_r - r_t) - \lambda r_t]dt + \sigma_r \sqrt{r_t} dZ_{tr}^Q,$$  \hspace{1cm} (11)

where $Z_{tr}^Q$ is a one-dimensional Brownian motion under the Q-measure.

**Assumption 2:** The intensity $h_t$ evolves according to the equation
\[ dh_t = \kappa_h (\theta_h - h_t)dt + \sigma_h \sqrt{h_t} dZ_{2t}, \]  
\[ \sigma \theta + =, (12) \]

where \( \kappa_h, \theta_h > 0 \), and \( 2\kappa \theta \geq \sigma_h^2 \). The instantaneous correlation between \( dZ_{1t} \) and \( dZ_{2t} \) is \( \rho dt \). Under the Q-measure, I assume that the process can be written as

\[ dh_t = [\kappa_h (\theta_h - h_t) - \lambda_h h_t]dt + \sigma_h \sqrt{h_t} dZ_{2t}^Q, \]  
\[ \sigma \lambda + - =, (13) \]

where \( Z_{2t}^Q \) is a one-dimensional Brownian motion under the Q-measure.

**Assumption 3:** The loss rate \( L_t \) is assumed to be a constant, \( L_t = L \).

Note that Assumption 3 is not necessary. The loss rate can be deterministic or even random. One of my objectives is to examine how different assumptions on the recovery affect the valuation of defaultable contingent claims. By making such a simplified assumption, I can concentrate on this issue.

**Assumption 4:** I assume that coupon is paid continuously, and the annualized coupon payment is \( c \).

The reason for us to assume continuous coupon payment instead of semi-annual coupon payment is that it is relatively easy to solve partial differential equations (PDE) using numerical methods.

In this paper, I rely on numerical methods to value several contingent claims. Specifically, alternating direction implicit (ADI) method is used to deal with this two-factor model. Although the CIR-type process used in this paper belongs to the “affine”

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5 For example, see Duffie and Dingleton (1999) and Skinner and Diaz (2000).
family, in which a closed-form solution can then be derived, most interest rate and default risk models are not of the “affine” type, therefore an analytical solution is unavailable. Moreover, numerical methods allow us to consider a wide range of models used in the literature. To be able to apply numerical methods, I need to find the PDEs under different recovery assumptions. This can be done by applying the Feynman-Kac representation (see the Appendix for details).

Recovery of face value (RFV)
According to equation (6) and the Feynman-Kac formula, $V^{\text{RFV}}(t,T,c,L)$ is the unique solution to the following PDE

$$\frac{1}{2} \sigma_r^2 r V_{rr} + \frac{1}{2} \sigma_h^2 h V_{hh} + [\kappa_r(\theta_r - r) - \lambda_r r] V_r + [\kappa_h(\theta_h - h) - \lambda_h h] V_h .$$

$$+ \rho \sigma_r \sigma_h \sqrt{h} V_{hr} - (r + h) V + c + (1 - L) h + V_t = 0 .$$

(14)

with the boundary condition

$$V(T,T,c,L) = 1 .$$

(15)

Recovery of market value (RMV)
According to equation (9) and the Feynman-Kac formula, $V^{\text{RMV}}(t,T,c,L)$ is the unique solution to the following PDE

$$\frac{1}{2} \sigma_r^2 r V_{rr} + \frac{1}{2} \sigma_h^2 h V_{hh} + [\kappa_r(\theta_r - r) - \lambda_r r] V_r + [\kappa_h(\theta_h - h) - \lambda_h h] V_h .$$

$$+ \rho \sigma_r \sigma_h \sqrt{h} V_{hr} - (r + hL) V + c + V_t = 0 .$$

(16)

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6 The affine term structural model is a class of models in which the yields to maturity are affine (constant-plus-linear) functions in some state variable vector $X_t$. Examples of the affine processes in the term-structure literature are the Gaussian (Vasicek, 1977) and square-root diffusion models (Cox, Ingersoll, and Ross, 1985). Detail discussions of affine models can be found in Dai and Singleton (2000), and Duan and Simonato (1999).
with the boundary condition

\[ V(T, T, c, L) = 1. \]

4 Valuation of defaultable, callable bonds

The majority of corporate bonds are callable. The call option gives the issuer the right to call the bond at a fixed call price any time before the bond maturity, after an initial “lock-out” period.\(^7\) The issuer may call back the bond under the following two situations: when the interest rate falls or when the credit quality of the issuer improves. In either case, the issuer calls the bond and replaces it with lower-cost debt. According to Buttler and Waldvogel (1996), there are three types of callable bonds:

1. European callable bonds: the issuer has the right to call the bond at only one date (typically the last coupon date before maturity).
2. American callable bonds: the issuer may call the bond at any time after an initial “lock-out” period.
3. Semi-American (Bermudan) callable bonds: the issuer has the right to call the bond at one of a set of pre-specified dates (usually coinciding with coupon dates) after a “lock-out” period.

In this section, I apply my pricing model to examine the valuation of defaultable callable bonds when both interest rates and intensity are stochastic. I assume that the issuer follows rule for calling bond so as to minimize the market value of that bond. This rule implies that the issuer will exercise the option to call in the bond at time \( \tau \) if and only if its market price, if not called, is higher than the strike price on the call. I thus have another boundary condition

\[ V(\tau, T, c, L) = \min(V(\tau, T, c, L), K_\tau), \]

\(^7\) The “lock-out” period is defined as the length of time from issuance until the first possible call date. The range of the “lock-out” period is from as short as a month to more than ten years.
where \( V(\tau, T, c, L) \) is the bond price at time \( \tau \), assuming that the bond has not defaulted by \( \tau \), and \( K_\tau \) is the call price at time \( \tau \).

Then the valuation equations (14) and (16), subject to equations (15) and (17), can be solved numerically by the ADI method, respectively.

Several papers in the literature have looked at this issue. However, because of the complexity of the default and call options, much of the existing work has treated interest rates as constant. From the perspective of my paper, the two most relevant papers are Acharya and Carpenter (2000) and D’halluin, Forsyth, Vetzal, and Labahn (2001).

Acharya and Carpenter consider both interest rates and default risk, but they employ the structural approach to model default risk. In addition, they do not take into account the notice period. In practice, however, most callable bonds require that the issuer provide an advance notice of a decision to exercise the embedded call. D’halluin et al. (2001) use a numerical PDE approach to price callable bonds with the notice period. However, they focus on only default-free contracts.

The valuation model applied in this paper incorporates all the following features: interest rate risk, default risk, correlation between these two risks, call provisions, optimal call policies, and the notice period. To clarify the interaction of the call and default options, I also look at several simpler counterparts to defaultable, callable bonds: a defaultable, non-callable bond, and a default-free non-callable bond with the same face value and maturity, and a default-free callable bond with the same face value, maturity and call scheme.

My benchmark is an American defaultable, callable bond without the notice period. I assume that the coupon is continuously paid and the RFV assumption is used. Table 1 gives a summary of the parameter values and call scheme. The parameter values for the interest rate process are \( \kappa_r = 0.55 \), \( \theta_r = 0.035 \), \( \sigma_r = 0.38 \), \( \lambda_r = -0.41 \), which are taken directly from D’halluin et al (2001). The parameter values for the intensity process are

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8 Examples include Merton (1974), Brennan and Schwartz (1977), and Black and Cox (1976), etc.
\( \kappa_h = 0.22, \ \theta_h = 0.0058, \ \sigma_h = 0.073, \text{ and } \lambda_h = -0.25 \). These values are chosen based on the work of Duffee (1999). In his paper, Duffee estimates these parameter values based on the credit ratings of the bonds’ issuers. A total of 161 firms are included and 5 different credit ratings are used to implement the estimation. For my purpose, I need only one set of parameter values that are applicable to all credit ratings, so I compute weighted average values of the parameters across different credit ratings. The recovery rate used throughout this paper is 0.4, which is suggested in Altman and Kishore (1996). The base case correlation between interest rates and intensity, \( \rho \), is zero. The defaultable, callable bond I consider here is a 20-year bond with a 10-year “lock-out” period.

**Sensitivity analysis to \( \rho \)**

The effect of stochastic interest rates and stochastic intensity on total spreads depends on the correlation between these two stochastic variables, \( \rho \). I consider the cases when \( \rho = -0.8, 0, \text{ and } 0.8 \). Figure 1 illustrates how total spreads of a 20-year defaultable, callable bond with a 10-year “lock-out” period change when the correlation \( \rho \) takes different values, given initial interest rate \( r_0 \). Two types of callable bonds are considered, one with low credit quality, the other with high credit quality, based on initial default intensity \( h_0 \).

Here is what I find from Figure 1:

- For a callable bond with given maturity, as the initial interest rate \( r_0 \) goes up, the total spread goes up. Moreover, the total spread goes up as \( \rho \) decreases. This result is consistent with Acharya and Carpenter (2000).¹⁰

The first part of the results can be explained intuitively. As the initial interest rate \( r_0 \) goes up, because of the nature of mean-reversion of the interest rate process, the probability of

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⁹ The total spread in our paper is defined as the difference between the par-coupon yield of defaultable, callable bond and the par-coupon yield of default-free non-callable bond with the same maturities. This definition is used to distinguish traditional definition of par-coupon credit spread of non-callable bonds.

¹⁰ Acharya and Carpenter (2000) claim that spreads move in the same direction as the correlation between interest rates and firm value. Since the higher the firm value, the less likely the firm defaults, it is equivalent to say that spreads widen as the correlation between interest rates and intensity decreases.
decrease in interest rates in the future is high, then the probability of calling the bond is high, so investors ask for higher spread to compensate for it.

- The effect of the correlation measure on spreads is greater on low quality bonds than on high quality bonds.

In Figure 2, I describe how term structures of par-coupon total spreads of defaultable, callable bonds move as $\rho$ changes. I find:

- For a short-term bond, the total spread increases as $\rho$ goes up; for a long-term bond, however, the total spread increases when $\rho$ decreases. Furthermore, the effect of $\rho$ on the total spreads is much more stronger for low quality bonds than for high quality bonds. For a 10-year bond, the effect of changing $\rho$ from –0.8 to 0.8 is approximately 67bps for low quality bonds, whereas the difference is only around 16bps for high quality bonds.

For comparison purposes, I also calculate spreads for defaultable, non-callable bonds with different $\rho$. The results are shown in Figure 3. I find:

- The spreads of defaultable, non-callable bonds are also quite sensitive to the value of $\rho$, and the effect of $\rho$ on spreads is stronger as the bond’s quality is lower. All these results are quite similar to those of defaultable, callable bonds. The effect of $\rho$ on spreads of non-callable bonds are consistent; that is, the total spread always increases as $\rho$ goes down, and this result is independent of the maturity.

Call premiums and credit spreads

People in risk management may be interested in answering the following questions:

1. Should investors require the same credit spreads for both callable and non-callable bonds?
2. Should investors require the same call premiums for both defaultable and default-free bonds?

Interestingly, the answers to these two questions are basically the same. There are two ways to calculate total spreads: one way is to calculate the sum of credit spreads of
callable bonds and call premiums of default-free bonds; the other way is to calculate the sum of credit spreads of non-callable bonds and call premiums of defaultable bonds. Then different credit spreads of callable and non-callable bonds imply different call premiums of defaultable and default-free bonds. Figure 4 shows that under assumption of zero correlation, the par-coupon credit spreads for callable and non-callable bonds are not the same for a 20-year bond, and are not even close at high interest rates. For example, when the interest rate is 15% per year, the par-coupon credit spreads of 20-year callable bonds are 77.59 basis points for high quality bonds and 472.30 basis points for low quality bonds; for non-callable bonds the spreads are 109.99 basis points for high quality and 516.18 basis points for low quality bond. Figures 5 and 6 show the cases with positive and negative $\rho$'s, respectively. My results show that investors indeed require different spreads for callable and non-callable bonds and different call premiums for defaultable and default-free bonds.

Comparison of total spreads under the RFV and RMV assumptions

Duffie and Singleton (1999) show that for non-callable bonds, the term structures of par-coupon yield spreads for RMV and RFV are rather similar even when the recovery rates in both cases are the same. To find out whether this result applies to defaultable, callable bonds as well, I calculate the term structures of par-coupon total spreads of defaultable, callable bonds under these two assumptions.

- The assumption of RMV gives a larger spread. The higher the initial value of intensity, $h_0$, the larger gap of the spread between these two assumptions. This result is consistent with the patterns that Duffie and Singleton (1999) find empirically for non-callable bonds.

- For defaultable, callable bonds, the term structures of par-coupon total spreads under RMV and under RFV are not quite the same, especially for longer-term, low quality bonds. For example, under the assumption of zero correlation, consider a 10-year defaultable, callable bond with a long-run mean equals 58 basis points, the total spread is 509.16 basis points under RMV, and 485.57 basis
points under RFV, the difference is 23.59 basis points, which is quite significant. Figure 7 shows the results.

For comparison purposes, I also calculate the term structure of par-coupon yield spreads for defaultable, non-callable bonds under both the RMV and RFV assumptions by using exactly the same parameter values.

- For defaultable, non-callable bonds, the two recovery assumptions give us rather similar term structures of par-coupon yield spreads for short-term and medium-term bonds. This result is quite consistent with Duffie and Singleton (1999) even though when I use different models and parameter values. For example, the gap for a 10-year defaultable, non-callable bond under these two different assumptions is only 15.37 basis points. For long-term bonds, however, the gap is quite noticeable. It is shown in Figure 8. Table 2 gives the comparison.

**The effect of the notice period on valuation of callable bonds**

In practice, most callable bonds require that the issuer provide an advance notice of its intention to exercise the embedded call. As noted by Bliss and Ronn (1998), the standard description of the optimal call policy for the issuer is no longer correct when the advance notice must be given. To find out the effect of notice periods on the valuation of callable bonds, in this section, I consider the valuation of a 20-year American callable bond with a 3-month notice period, everything else being the same as in the case without the notice period. My calling rule follows Jordan and Jorgensen (1996) where a bond is called when the price of the callable bond exceeds the price of a 3-month Treasury bill on the notification date. More specifically, since my call price is not a single value but a scheme, the face values of these 3-month Treasury bills change as the call price changes. I calculate the total spreads of the callable bond with the notice period and compare the results to the bond without-notice-period. Table 3 shows the results of the comparison. Several conclusions can be drawn from Table 3:

- As the initial interest rate \( r_0 \) goes up, the total spread required by investors also goes up regardless of the initial quality of the bond.
• For any given $r_0$, the total spread of callable bond without the notice period is higher than that of callable bond with the notice period. The above result is quite intuitive as that the notice period is valuable to investors, and they would like to pay more for this option. The total spread for the bond with the notice period thus is lower than the one without the notice period.

• Although the total spreads are different by adding the notice period, the change is not significant. For example, consider a 20-year defaultable, callable bond with a long-run mean equals 58 basis points and $\rho = 0$, the total spread of this bond with a 3-month notice period is 226.98 basis points when $r_0 = 15\%$ and $h_0 = 50$ basis points; the total spread of the same bond without the notice period is 231.89 basis points. This can be seen in Figure 9.

5 Valuation of default swaps
Credit derivatives are contracts that transfer an asset’s risk and return from one counterpart to another without transferring ownership of the underlying asset. There are four major types of credit derivatives: default swaps, total return swaps, credit spread put options, and credit linked notes.\textsuperscript{11} The global market for credit derivatives is still quite small compared with other derivatives markets. It represents only 1% of the global derivatives market, but it is growing rapidly. Figure 10 shows the increased trading volume of credit derivatives. The exponential growth as seen in Figure 10 has generated significant interests in the fair valuation of credit derivatives in both the academic and practitioner communities.

Among these four types of credit derivatives, the default swap is the most common credit derivative. According to the British Bankers Association (BBA) 2001 survey, 40% of the market notional amounts outstanding come from credit default swaps. A default swap is a contract that provides insurance against the risk of a default by a particular company. The protection seller pays the protection buyer a given contingent amount if there is a credit

\textsuperscript{11} A good introduction on credit derivatives can be found in Bomfim (2001).
event, such as default. In return, the protection buyer makes periodic premium payments to the protection seller until the time of the credit event, or the maturity date of the credit swap, whichever is first. In the case of a credit event the default swap can be settled by either physical delivery or in cash. In a physically settled swap, the protection seller receives the underlying and pays the face value to the protection buyer. In a cash settled swap, the protection seller pays the difference between par and the recovery value of the underlying.

Because of the popularity of default swaps, there are quite many papers working on the valuation of swaps. Most of them assume that the credit and the market risk are statistically independent. Examples are given in Bomfim (2001), Dwljanedis and Lagnado (2000), and Hull and White (2000). Based on this simplified assumption, these papers develop models for pricing default swaps. One noticeable exception is Jarrow and Yildirim (2002), who relax this independence assumption and still provide a simple analytic formula for the valuation of default swaps. Specifically, to obtain their simple while realistic empirical formulation of the model, Jarrow and Yildirim assume that the economy is Markovian in a single state variable – the spot interest rate, and that the intensity is a linear function of the spot rate, which is used to incorporate the correlation between the credit and the market risks in their model. An analytic expression is then derived in the context of a reduced-form credit risk model.

My approach, in some ways, follows Jarrow and Yildirim’s (2002) work. I also use the reduced-form credit risk model and incorporate the correlation between intensity and spot rate in my model. However, one major difference of my work from theirs is that, I relax the assumption of a linear relationship between the credit and the market risks; instead, I model stochastic processes for intensity and spot rates separately. The correlation is specified in the random part. The numerical procedure is then used in my paper to value default swaps.

12 Other commonly used credit events that are defined by the International Swaps and Derivatives Association (ISDA) are: failure to pay, bankruptcy, cross-default, restructuring, cross-acceleration, repudiation, merger, regulatory suspension, and downgrading.
The pricing of default swaps is fundamentally linked to three factors: (1) the credit risk of the reference entity, (2) the expected recovery rate associated with the reference entity, (3) the credit risk of the protection seller. The first and third factors highlight the two types of risk faced by the protection buyer: issuer default risk, and counterparty default risk. A fourth factor may also affect the pricing of default swaps: the default correlation between the reference entity and the protection seller. However, such a joint event is typically much less likely than a default by the reference entity or the protection seller alone, so I do not consider the joint default in this paper. In addition, I also assume that there is no counterparty default risk, so I do not need to consider the pricing impact of default by the protection seller.

There are two pricing issues associated with a default swap:

1. At the beginning of the contract, the standard default swap involves no exchange of cash flows, and therefore has zero market value. One must determine the annuity premium for which the market value of the default swap is indeed zero. This premium is sometimes called the default swap spread.

2. After origination, given the annuity premium, one must determine the current market value of the default swap, which is generally non-zero.

In this section, I focus on the valuation of the default swap spread.

According to the discussion above, a default swap typically has a zero market value when it is set up, and thus pricing such a contract is equivalent to finding the value of the default swap spread, \( s \), that makes the expected present value of payments made by the buyer have the same value as the expected present value of payments received by the buyer in the case of default.

**Assumption 5:** I assume that the default swap spread, \( s \), is paid continuously.
This assumption simplifies my procedure in the sense that I do not need calculate the accrued portion of payment in the case when the credit event happens between the payment dates.

Under the RFV assumption of recovery, the expected present value of payments made by the buyer is given by

\[
E_t^Q \left[ \int_t^T e^{-\int_t^u \phi_t \, du} \right] = E_t^Q \left[ \int_t^T e^{-\Phi(u)} \, du \right], \tag{18}
\]

where \( \phi_t = r_t + h_t \), and \( \Phi(\tau) = \int_0^\tau \phi_u \, du \).

The expected present value of payments received by the buyer in case of default is

\[
E_t^Q \left[ e^{-\int_t^\tau \phi_u \, du} \cdot L \cdot 1_{\{\tau \leq T\}} \right] = E_t^Q \left[ \int_t^T L \cdot e^{-\Phi(u)} \cdot h_u \, du \right]. \tag{19}
\]

The credit default spread, \( s \), is the value that makes expressions in equations (18) and (19) equal

\[
s = L \cdot \frac{E_t^Q \left[ \int_t^\tau e^{-\Phi(u)} \cdot h_u \, du \right]}{E_t^Q \left[ \int_t^T e^{-\Phi(u)} \, du \right]}. \tag{20}
\]

In this subsection, I implement numerical methods to estimate the credit default spread, \( s \), assuming both the spot rate and the intensity processes are stochastic. My benchmark is a 10-year credit default swap contract with zero correlation (\( \rho = 0 \)) between interest rates and intensity. To evaluate the effect of the correlation on the valuation of the credit default spread, I also consider two other cases: \( \rho > 0 \) and \( \rho < 0 \). In addition, I explore the
term structure of the credit default spread and trace the effect of changing $\rho$ on the shape of the term structure of the credit default spread. The values of model parameters are summarized in Table 1. Through examining Figure 11, I have the following findings:

- The choice of the initial interest rate $r_0$ has no significant effect on the valuation of the credit default spread given the initial intensity $h_0$, and this result is not sensitive to the choice of $\rho$, especially for the high quality reference entities. For example, consider a case where $\rho$ is zero and $h_0 = 50$ basis points, the credit default spread is 68.36 basis points when $r_0 = 0.0\%$, and 65.84 basis points when $r_0 = 15.0\%$.

- As $\rho$ goes up, the credit default spread goes down.

Figures 12 and 13 show the credit default spread term structure for different initial credit qualities. I find:

- For short-term swaps, the change of $\rho$ has little effect on the valuation of the credit default spread, regardless of the credit quality. For example, for a 2-year swaps with high quality reference entities, the credit default spread is 39.38 basis points when $\rho = -0.8$, 38.45 basis points when $\rho = 0.0$, and 37.28 basis points when $\rho = 0.8$. For a 2-year swaps with low quality reference entities, the credit default spread is 380.15 basis points when $\rho = -0.8$, 376.84 basis points when $\rho = 0.0$, and 373.33 basis points when $\rho = 0.8$. The differences are less than 1 basis point for high quality reference entities and less than 4 basis points for low quality reference entities.

- For longer-term swaps, the values of credit default spreads differ significantly as $\rho$ changes. For example, consider a 10-year swaps with low quality reference entities, the credit default spread is 449.41 basis points when $\rho = -0.8$, 412.12 basis points when $\rho = 0.0$, and 376.49 basis points when $\rho = 0.8$. The differences are more than 30 basis points, which are quite noticeable.
• The percentage changes of credit default spreads are not symmetric for a high quality reference entity as ρ changes from a negative value to zero and from zero to a positive value, especially for long-term swaps. For example, the percentage change of the credit default spread for a 10-year swaps with high quality reference entities is 18.7% when ρ changes from a negative value to zero; and the percentage change increases to 31.1% as ρ changes from zero to a positive number.

6 Conclusion

This paper studies the valuation of defaultable, callable bonds and credit default swaps when both interest rates and default intensity are stochastic, and the correlation between these two variables is nontrivial. I determine the prices of these two contingent claims and examine the effect of different correlation on valuation.

For the defaultable, callable bond, my main findings are as follows. First, the total spread has negative relationship with the correlation, ρ, and the magnitude of this correlation effect changes with the initial credit quality of the bond. Second, the influence of changing ρ on the term structures of par-coupon total spreads of defaultable, callable bond is quite different from its defaultable, non-callable counterpart. Third, investors require different spreads for callable and non-callable bonds and different call premiums for defaultable and default-free bonds. Fourth, different assumptions on the recovery make noticeable differences on the total spreads of medium-term and long-term defaultable, callable bonds. Fifth, adding the notice period makes no significant change of the total spread.

For the credit default swaps, I find that first, the choice of interest rate, \( r_0 \), has no significant effect on the valuation of the credit default spread. Second, as the correlation ρ goes up, the credit default spread goes down. Third, the effect of changing ρ on the absolute changes of credit default spreads is stronger for low quality reference entities. Fourth, the change of ρ has little effect on the valuation of the credit default swaps for
short-term swaps, for longer-term swaps, the values of credit default spreads diverge drastically as $\rho$ changes.
Table 1. Summary of model parameters

<table>
<thead>
<tr>
<th>Maturity T</th>
<th>Interest rate</th>
<th>Intensity</th>
<th>Year from T</th>
<th>Call price</th>
</tr>
</thead>
<tbody>
<tr>
<td>20 years</td>
<td>$\kappa$</td>
<td>0.55</td>
<td>0.22</td>
<td>1-5 $100</td>
</tr>
<tr>
<td>Coupon C</td>
<td>$\theta$</td>
<td>0.035</td>
<td>0.0058</td>
<td>6 $100.5</td>
</tr>
<tr>
<td>Principal</td>
<td>$\sigma$</td>
<td>0.38</td>
<td>0.073</td>
<td>7 $101</td>
</tr>
<tr>
<td>Loss rate</td>
<td>$\lambda$</td>
<td>-0.41</td>
<td>-0.25</td>
<td>8 $101.5</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>9 $102</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>10 $102.5</td>
</tr>
</tbody>
</table>

This table gives my choice of parameter values used in the paper. Input data used for the models considered. The interest rate process parameter values are from D'halluin et al. (2001), and the intensity process parameter values are chosen based on Duffee (1999). Note that the years for the call prices run backwards in time, so for example the bond is callable at a price of $100.5 six years before maturity. Coupon payments are on an annual basis.

Table 2. Spread differences under the RMV and RFV assumptions

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Type of bonds</th>
<th>H_mean = 58bps, $\rho = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$h_0 = 50$bps</td>
</tr>
<tr>
<td>5 year</td>
<td>Callable</td>
<td>1.52</td>
</tr>
<tr>
<td></td>
<td>Non-callable</td>
<td>0.61</td>
</tr>
<tr>
<td>10 year</td>
<td>Callable</td>
<td>3.53</td>
</tr>
<tr>
<td></td>
<td>Non-callable</td>
<td>2.19</td>
</tr>
<tr>
<td>20 year</td>
<td>Callable</td>
<td>9.96</td>
</tr>
<tr>
<td></td>
<td>Non-callable</td>
<td>7.82</td>
</tr>
</tbody>
</table>

The spread difference is given in basis points. I consider 5-, 10-, and 20-year callable and non-callable bonds with different recovery assumptions. Two types of callable bonds are considered here: one with low quality ($h_0=600$bps), the other with high quality ($h_0=50$bps). The parameter values are given in Table 1.
Table 3. Comparison of total spreads of callable bonds with and without the notice period

<table>
<thead>
<tr>
<th>$r_0$ (%)</th>
<th>With the notice period</th>
<th>Without the notice period</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$h_0 = 50$bps</td>
<td>$h_0 = 600$bps</td>
</tr>
<tr>
<td>0</td>
<td>98.35</td>
<td>402.20</td>
</tr>
<tr>
<td>2.5</td>
<td>111.92</td>
<td>433.27</td>
</tr>
<tr>
<td>5.0</td>
<td>130.53</td>
<td>466.80</td>
</tr>
<tr>
<td>7.5</td>
<td>151.21</td>
<td>502.61</td>
</tr>
<tr>
<td>10.0</td>
<td>173.92</td>
<td>540.88</td>
</tr>
<tr>
<td>12.5</td>
<td>199.14</td>
<td>581.31</td>
</tr>
<tr>
<td>15.0</td>
<td>226.98</td>
<td>623.36</td>
</tr>
</tbody>
</table>

The total spreads are given in basis points. I calculate the total spreads of a 20-year American callable bond with 10-year lockout period with and without the notice period. The notice period is assumed to be 3 months. Two types of callable bonds are considered here: one with low credit quality ($h_0=600$bps), the other with high credit quality ($h_0=50$bps). The correlation, $\rho$, is assumed to be zero. The parameter values are given in Table 1.
Figure 1. Total spreads of a 20-year defaulatable, callable bond with a 10-year “lock-out” period. The recovery rate is 0.4 and the call scheme is shown in Table 1. Two types of callable bonds are considered here: one with low credit quality ($h_0 = 600$bps), the other with high credit quality ($h_0 = 50$bps). Three different values of the correlation variable, $\rho$, are considered: -0.8, 0, and 0.8.
Figure 2. Term structures of par-coupon total spreads of defaultable, callable bonds under RFV. Two types of callable bonds are considered here: one with low credit quality ($h_0 = 600$bps), the other with high credit quality ($h_0 = 50$bps). Three different values of the correlation variable, $\rho$, are used: -0.8, 0, and 0.8. The values of model parameters are given in Table 1. Note that to simplify my procedure, I use a single call price, $101.00, instead of a call scheme, and I assume that the first call dates of all maturities are one half of their maturities, respectively. The initial interest rate, $r_0$, is chosen to be equal to 3.75%.
Figure 3. Term structures of par-coupon total spreads of defaultable, non-callable bonds under RFV.
Two types of callable bonds are considered here: one with low credit quality ($h_0 = 600$bps), the other with high credit quality ($h_0 = 50$bps). Three different values of the correlation variable, $\rho$, are used: -0.8, 0, and 0.8. The values of model parameters are contained in Table 1.
Figure 4. Par-coupon credit spreads of 20-year callable and non-callable bonds. These two bonds have the same face value, maturity. For the callable bond, the “lock-out” period is 10 years and the call scheme is shown in Table 1. The recovery rate is 0.4 in both cases. Two types of callable bonds are considered here: one with low credit quality ($h_0 = 600$ bps), the other with high credit quality ($h_0 = 50$ bps). The correlation parameter, $\rho$, is assumed to be zero.
Figure 5. Par-coupon credit spreads of 20-year callable and non-callable bonds. These two bonds have the same face value, maturity and issuer. For the callable bond, the “lock-out” period is 10 years and the call scheme is shown in Table 1. The recovery rate is 0.4 in both cases. Two types of callable bonds are considered here: one with low credit quality ($h_0 = 600$bps), the other with high credit quality ($h_0 = 50$bps). The correlation parameter, $\rho$, is assumed to be 0.8.
Figure 6. Par-coupon credit spreads of 20-year callable and non-callable bonds. These two bonds have the same face value, maturity and issuer. For the callable bond, the “lock-out” period is 10 years and the call scheme is shown in Table 1. The recovery rate is 0.4 in both cases. Two types of callable bonds are considered here: one with low credit quality ($h_0 = 600$bps), the other with high credit quality ($h_0 = 50$bps). The correlation parameter, $\rho$, is assumed to be -0.8.
Figure 7. Term structures of par-coupon total spreads of a defaultable, callable bond under RMV and RFV. Two types of callable bonds are considered here: one with low credit quality ($h_0 = 600$bps), the other with high credit quality ($h_0 = 50$bps). The correlation parameter, $\rho$, is assumed to be zero. The values of model parameters are given in Table 1. Note that to simplify my procedure, I use a single call price, $\$101.00$, instead of a call scheme, and I assume that the first call dates of all maturities are one half of their respective maturities. The initial interest rate, $r_0$, is chosen to be equal to 3.75%. The recovery rate is 0.4 in both cases.
Figure 8. Term structures of par-coupon total spreads of defaultable, non-callable bonds under RMV and RFV. Two types of callable bonds are considered here: one with low credit quality \((h_0 = 600\text{bps})\), the other with high credit quality \((h_0 = 50\text{bps})\). The correlation parameter, \(\rho\), is assumed to be zero. The values of model parameters are given in Table 1. The recovery rate is 0.4 in both cases.
Figure 9. Par-coupon total spreads of 20-year defaultable, callable bonds with and without the notice period. The notice period is 3 months. The recovery rate is 0.4 in both cases. Two types of callable bonds are considered here: one with low credit quality ($h_0 = 600$bps), the other with high credit quality ($h_0 = 50$bps). The correlation parameter, $\rho$, is assumed to be zero.
Figure 10. Credit derivatives notional volumes.
Figure 11. Credit default spreads of 10-year contract under RFV. The recovery rate is 0.4. Two types of reference entities are considered here: one with low credit quality ($h_0 = 600$bps), the other with high credit quality ($h_0 = 50$bps). Three different values of the correlation variable, $\rho$, are considered: -0.8, 0, and 0.8.
Figure 12. Term structures of credit default spreads under RFV. The values of model parameters are given in Table 1. The recovery rate is 0.4. I consider only low credit quality reference entities ($h_0 = 600$ bps). Three different values of the correlation variable, $\rho$, are considered: -0.8, 0, and 0.8.
Figure 13. Term structures of credit default spreads under RFV. The values of model parameters are given in Table 1. The recovery rate is 0.4. I consider only high credit quality entities ($h_0 = 50$ bps). Three different values of the correlation variable, $\rho$, are considered: -0.8, 0, and 0.8.
Appendix: The Feynman-Kac formula

Let $X_t$ be the following diffusion process

$$dX_t = x + \int_t^T b(X_u, u)du + \int_t^T \sigma(X_u, u)dW_u.$$ 

Proposition: Feynman-Kac formula if $b$ and $\sigma$ satisfy the Lipschitz condition, if the real-valued functions $f, g$ and $\rho$ satisfy the Lipschitz condition on $\mathbb{R}^K \times [0,T]$ for a scalar $T > 0$, and if the functions $b, \sigma, f, g, \rho, b_x, \sigma_x, u_x, f_x, \rho_x, b_{xx}, \sigma_{xx}, u_{xx}, f_{xx}$ are continuous and satisfy the growth conditions, then the (twice continuously-differentiable) function $V: \mathbb{R}^K \times [0,T] \to \mathbb{R}$ defined by

$$V(x,t) = E[\int_t^T e^{-\phi(s)} f(X_u, u)du + e^{-\phi(T)} g(X_T, T)],$$

where $\phi$ is the discount factor and

$$\phi(s) = \int_s^T \rho(X_u, u)du$$

is the unique solution to the following partial differential equation

$$DV(x,t) - \rho(x,t)V(x,t) + f(x,t) = 0 \in \mathbb{R}^K \times [0,T],$$

with limit condition

$$V(x,T) = g(x,T), x \in \mathbb{R}^K,$$

where

$$DV(x,t) = V_x(x,t) + V_t(x,t)b(x,t) + \frac{1}{2} tr[\sigma^T(x,t)V_{xx}(x,t)\sigma(x,t)].$$
References


Bomfim, A.N., 2000, Understanding Credit Derivatives and their Potential to Synthesize Riskless Assets, Federal Reserve Board.


