Basket Options on Heterogeneous Underlying Assets

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Abstract

In this article, we make the design of a basket option on commodity prices with stochastic convenience yields, exchange rates and zero-coupon bonds. We carry out Monte Carlo simulations to value this option. We use maximum likelihood method to estimate the different parameters of the theoretical model proposed as well as the correlations between these variables. As done in Duan (1994), we apply maximum likelihood approach adapted to unobservable variables to estimate the temporal series of the convenience yield as well as its parameters. Monte Carlo studies are conducted to examine the performance of the maximum likelihood technique in finite samples of simulated data.

Key Words: Basket options, Maximum likelihood, Option pricing, Monte Carlo simulation.

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1 Introduction

Basket options are now an increasingly popular instrument for hedging risky assets. First developed in the 1980s, these new products were not often used by investors as a key tool for reducing risk until the 1990s. A basket option combines options on a collection of assets whose final payoff will depend on the value of the whole portfolio. Since basket options are often traded over-the-counter, their composition depends on the specific needs of corporate treasurers. In this paper, we study a particular type of basket option based on commodity prices, exchange rates, and zero-coupon bonds. This choice is motivated by the fact that many non-financial institutions such as gold-mining firms, energy companies or airlines companies look for the most efficient way to hedge the various financial risks they must face. Take, for example, a gold-mining firm; we find it is exposed to different types of risk: commodity-price risks which include fluctuations in the price of its primary product, gold, as well as of its by-products such as silver and copper, currency-exchange risks, and interest-rate risks. Since many Canadian gold-mining firms sell their products in the United States and other countries, they have to convert their non-Canadian revenues and costs into Canadian dollars for tax and accounting purposes and are thereby exposed to the exchange-rate risk. Their interest-rate-risk exposure is primarily related to changes in the fair value of fixed-rate debt obligations and to the borrowing costs on variable-rate obligations. Clearly, all these markets have active and liquid options exchanges and so each of these risks could be hedged separately. Nevertheless, it seems more interesting for these firms to use a basket option as a single hedge against these exposures in order to benefit from possible correlations between these risks. In fact, as the basket volatility is the weighted sum of the volatilities of each component it contains, when the underlying assets are negatively correlated, the portfolio is partially diversified and this effectively reduces volatility and therefore the basket value.

However, in spite of the apparent simplicity of this financial instrument, basket options are more complicated to evaluate than standard options because their value depends on the weighted sum of multiple correlated assets. The main theoretical reason for this difficulty is that one has not yet derived a closed-form expression for the probability density function of the underlying assets that would serve as an adequate analytical tool in assessing the basket-option price. Several approaches have been developed in the literature to assess the price of basket options. We can group them into two categories: general numerical methods and analytic approximations that produce closed-form expressions. In the numerical techniques category, Rubinstein (1994) has developed a multivariate binomial tree, but this is not a very efficient way to price basket options, as the number of nodes grows exponentially with the number of assets in the basket. Barraquand (1995) and Pellizzari (1998) have used Monte Carlo simulations with different variance-reduction techniques. The former used quadratic resampling and the latter developed two control variates based, respectively, on unconditional and conditional expectations of assets. More recently, Dahl (1999) and Dahl and Benth (2001) have proposed the use of Quasi-Monte-Carlo simulations to price basket options. Several other authors have tried to find analytical solutions to basket options. Indeed, Gentle (1993) used Vorst’s technique for pricing an average-rate option. He approximated the weighted sum of the basket’s underlying assets with the respective geometric sum. Huynh (1994) developed Gentle’s (1993) methodology by using an Edgeworth series expansion around a known
lognormal distribution to describe the distribution of the basket’s underlying assets. Milevsky and Posner (1998) derived a closed-form expression for the price of the basket option by assuming that the sum of correlated lognormal random variables can be closely approximated by a reciprocal gamma distribution. Posner and Milevsky (1998) approximated the true distribution function by an approximate one, known in the statistical literature as Johnson (1949) family, for which the first four moments are identical. Finally, Flamouris and Giamouridis (2001) have evaluated basket options using a Bernoulli jump-diffusion stochastic process for the underlying asset. They have developed a closed-form solution by assuming that the sum of lognormal variables is also a lognormal variable.

The different articles mentioned above all assume that the basket is homogeneous, meaning that all the underlying assets would be identical and follow the same kind of stochastic process. The main contribution of this paper consists in considering a basket option on multiple underlying assets which are intrinsically different. We, in fact combine, in the same basket, commodity prices, exchange rates, and zero-coupon bonds. This produces a basket options where some underlying assets are log-normally distributed and others are Gaussian and, though it does complicate the pricing procedure, this mix of distributions is more consistent with observed data. It is important to point out that none of the studies mentioned in the literature has yet focussed on this kind of mixed basket. We have also supposed that the commodity price and the convenience yield share the same source of risk, which allows us to work with a complete market and, so, a single price for the basket option. Finally, our paper presents all the aspects of basket options such as modelisation, basket-option pricing and estimation of parameters. Our contribution can be very useful, especially for practitioners who use this kind of product for hedging.

The objective of this paper consists in developing a theoretical model for the valuation of a basket option under the equivalent martingale measure. Because our model depends on several underlying assets with different stochastic processes we do not obtain a closed-form solution for the price of the basket option. Hence, we carry out a Monte Carlo simulation in order to value the basket. Concerning the empirical implementation of the basket-pricing model, one of the main difficulties is that some variables—such as the convenience yield and the market prices of risks related to the domestic and foreign zero-coupon bonds—are not observable. A technique well suited to deal with situations where variables, though unobservable, are known to be generated by a Markov process is the maximum-likelihood method. We apply this technique to estimate all the parameters of the pricing model as well as the correlations between the underlying assets composing the basket. This estimation procedure is implemented empirically on simulated data, and its performance is analyzed using a Monte Carlo study. To our knowledge, the maximum-likelihood estimation method has not yet been applied to models for basket options. Our results show that all the parameters are reasonably well estimated except those defining the convenience-yield variable for which we supposed a mean reverting process.

The remainder of the article is organized as follows. In the next section, we will describe in details the risk-neutral pricing model of a basket option. Section III will examine the Monte Carlo simulation for option valuation. In Section IV we will derive the likelihood function for observable and unobservable variables and report the empirical estimates of the model. Section V concludes.
2 The model

In this section we develop a model for the valuation of a basket option on commodity prices, exchange rates and zero-coupon bonds. Various models have been developed in the literature to price commodity, exchange rate and zero-coupon bond derivatives. Concerning commodity derivatives, Gibson and Schwartz (1990) developed a two-factor model for pricing financial contingent claims. The first factor is the commodity spot price and the second is the instantaneous convenience yield determined by a mean reverting process. Schwartz (1997) extended this model by introducing stochastic interest rates. Miltersen and Schwartz (1998) (MS hereafter) also developed a model to value options on commodity futures; their model assumes both stochastic convenience yield and stochastic interest rates. But unlike Schwartz (1997), MS obtain a closed-form solution to evaluate options on futures and forward prices. Finally, Hilliard and Reis (1998) introduced jumps in the spot price in addition to stochastic convenience yield and stochastic interest rates. In this paper, we suppose, as do Gibson and Schwartz (1990), that the commodity price may be modelised by a geometrical Brownian motion with a stochastic convenience yield. The choice of the geometric Brownian motion is motivated by the fact that gold prices follow a lognormal distribution. In fact, as in Gibson and Schwartz (1990), we regress the logarithm of gold-price ratios on their lagged values; the results support our hypothesis and show no mean reverting tendency (See details in Appendix I). Our specification of the convenience yield’s stochastic process is motivated by Fama and French’s finding (1988) that the metal’s convenience yield shows a mean-reverting tendency.

Regarding exchange rates, Biger and Hull (1983) and Garman and Kholhagen (1983) derived a closed-form solution for the value of exchange rate options. They suppose that the exchange rate follows a geometric Brownian motion with constant volatility. However, Tucker (1985), Shastri and Tandon (1986), Bodurtha and Courtadon (1987) and Melino and Turnbull (1987) show that the Black and Scholes modified model performs poorly owing to the fact that theoretical prices differ from market prices. The model exhibits a strike-price bias as well as a time-to-maturity bias. This poor performance can be explained by the hypothesis attributing constant volatility and lognormality to exchange rates. Various contributions have, in fact, shown that: i) the distribution of exchange rate is leptokurtic; ii) its volatility is not constant; and iii) its skewness is not equal to zero. Hence, poor specification of the underlying asset’s distribution can affect the performance of the model through the probability density function. Chesney and Scott (1989) relax the constant-volatility hypothesis by assuming a mean-reverting process for the logarithm of the volatility. They also suppose that the correlation between exchange rate and volatility is equal to zero. Melino and Turnbull (1990) developed a model with stochastic volatility to value American options on exchange rates. But contrary to Chesney and Scott (1989), they suppose that the correlation between volatility and exchange rates is different from zero. Introducing stochastic volatility certainly does significantly improve the model’s performance, but it also complicates the option-valuation procedure. Since volatility is not a tradable asset, this will lead to an incomplete market. And since volatility is not observable, estimation of the parameters describing its behavior is rendered more complicated. To avoid these problems in this paper, we suppose - as do Biger and

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1 We have to note that our model is applied essentially to gold mining firms. However, it can be extended to any firm that faces commodity price risk, exchange rate risk and interest rate risk.
Hull (1983) and Garman and Kholhagen (1983) - that the exchange rate may be modelised by a geometric Brownian motion with constant volatility.

Finally, we assume that both domestic and foreign zero-coupon bonds follow the single-factor model of Heath, Jarrow and Morton (1992) (HJM hereafter), which takes the initial forward-interest-rate curve as given and derives the drift of the forward-interest-rate process under the risk-neutral measure consistent with no arbitrage. Our choice of the HJM approach is justified by the fact that it presents many advantages over the earlier approaches adopted by Vasicek (1977), Brennan and Schwartz ((1979), (1982)) and Cox and al (1985). First, the model’s construction matches the current term structure. Second, the model requires no assumptions about investor preferences. Finally, it offers a parsimonious representation of the market dynamics and requires only one specification, this being the form of the forward-rate volatility function. However, empirical implementation of the HJM model is quite difficult, because its overall computational complexity is increased by a non-Markovian evolution of the instantaneous spot-rate which requires that the entire history of the term structure must be carried forward. To overcome this difficulty we suppose a HJM model with constant volatility. This specification allows us to estimate the model’s parameters even if the spot rate is non Markovian and without the need to assume and estimate any functional form for the market price of interest rate risk. The HJM model with constant volatility is also Gaussian, which means that the joint distribution function of the components is easy to implement. A one drawback is that the interest rates may be negative.

2.1 Notation

The model is

\[ \begin{align*}
    dS_t &= \alpha_S S_t \ dt + \sigma_S S_t \ dW_t^{(1)}, \\
    d\delta_t &= \kappa (\theta - \delta_t) \ dt + \sigma_\delta \ dW_t^{(1)}, \\
    dC_t &= \alpha_C C_t \ dt + \sigma_C C_t \ dW_t^{(2)}, \\
    dD_t &= r_t D_t \ dt, \\
    dF_t &= u_t F_t \ dt, \\
    dP(t, T) &= P(t, T) \left[ (r_t - \sigma_f \gamma_f (T - t) - \frac{1}{2} \sigma_f^2 (T - t)^2) \ dt - \sigma_f (T - t) \ dW_t^{(3)} \right], \\
    dK(t, T) &= K(t, T) \left[ (u_t + \sigma_g (\sigma_C - \lambda_t) (T - t) - \frac{1}{2} \sigma_g^2 (T - t)^2) \ dt - \sigma_g (T - t) \ dW_t^{(4)} \right],
\end{align*} \] (1)

where \( S \) is the price of a commodity expressed in a domestic currency; \( \delta \) is its stochastic convenience yield (continuously compounded); \( C \) represents the exchange rate (the value in domestic currency of one unit of the foreign currency); \( r \) and \( u \) are respectively the domestic and the foreign instantaneous spot rates; \( P \) and \( K \) are respectively the domestic and foreign zero-coupon bonds; and, finally, \( D \) and \( F \) stand, respectively, for the domestic and foreign bank accounts. The four-dimensional
Brownian motion \( \left\{ \left( W_t^{(1)} W_t^{(2)} W_t^{(3)} W_t^{(4)} \right)' : 0 \leq t \leq T \right\} \) is constructed on the filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, P)\) with the correlation structure

\[
\text{Corr}_P \left( W_t^{(i)}, W_t^{(j)} \right) = \rho_{ij}, \text{ for any } i, j = \{1, 2, 3, 4\} \text{ and any } 0 \leq t \leq T.
\]

The deterministic parameters \(\alpha_S, \sigma_S, \kappa, \theta, \sigma_\delta, \alpha_C, \sigma_C, \sigma_f, \sigma_g, \gamma_t, \text{ and } \lambda_t\) are unknown and need to be estimated.

When we impose a stochastic process for the convenience yield, the principle of arbitrage makes it impossible to obtain a unique price for any commodity contingent claim. This is due to the fact that there are more sources of uncertainties than there are tradable assets in the market. Consequently, the risk associated with this convenience yield cannot be hedged. Most researchers have, therefore, introduced a market-price of risk for the convenience yield in accordance with the equilibrium theory of Cox, Ingersoll and Ross (1985). The presence of a stochastic convenience yield introduces a couple of technical problems: (i) since it is not an observable variable, estimation of its parameters becomes more complicated and (ii) since it is not a tradable asset, it will lead to an incomplete market unless we impose additional restrictions. To solve the second problem, we modify the Gibson and Schwartz (1990) model, by considering that the commodity price (1a) shares the same source of risk as its convenience yield (1b). This simplification may be justified by the fact that commodity return is strongly and positively correlated with the convenience yield.

This positive correlation is induced by the level of inventories. When the commodity’s inventories decrease due to scarcities, its spot price should increase and the convenience yield should also increase since futures prices will not increase as much as spot prices (Brennan (1990)). In this way, we will be able to have a stochastic convenience yield without introducing an additional source of risk. This avoids us from having to estimate the market price of risk associated with the stochastic convenience yield, which is not observable.

**Theorem 1** The risk neutral measure \(Q\) exists and is unique. Moreover on \((\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, Q)\), the model is

\[
\begin{align*}
    dS_t & = (r_t - \delta_t) S_t \, dt + \sigma_S S_t \, d\hat{W}_t^{(1)}, \\
    d\delta_t & = a(m - \delta_t) \, dt + \sigma_\delta \, d\hat{W}_t^{(1)}, \\
    dC_t & = (r_t - u_t) C_t \, dt + \sigma_C C_t \, d\hat{W}_t^{(2)}, \\
    dP(t, T) & = P(t, T) \left[ r_t \, dt - \sigma_f(T - t) \, d\hat{W}_t^{(3)} \right], \\
    dK(t, T) & = K(t, T) \left[ [u_t + \sigma_g \sigma_C(T - t)] \, dt - \sigma_g(T - t) \, d\hat{W}_t^{(4)} \right],
\end{align*}
\]

where

\[
a = \left( \kappa + \frac{\sigma_\delta}{\sigma_S} \right), \quad m = \frac{1}{a} \left[ \kappa \theta - \frac{\sigma_\delta}{\sigma_S} (\alpha_S + r_t) \right].
\]

The existence and the uniqueness of the risk-neutral measure \(Q\) guarantee that the market is arbitrage free and complete. The strong solution of the system of equations (2) is given respectively by,
\[
S_T = S_t \exp \left[ \int_t^T f(0, u) \, du - (m + \frac{1}{2} \sigma_S^2) (T - t) + (m - \delta_t) \left( \frac{1-e^{-a(T-t)}}{a} \right) \right. \\
\left. + \frac{1}{2} \sigma_f^2 (T^3 - t^3) + \sigma_f \int_t^T d \tilde{W}_u^{(3)} + \int_t^T \left\{ \sigma_S - \sigma_f \left( \frac{1-e^{-a(T-u)}}{a} \right) \right\} d \tilde{W}_u^{(1)} \right] 
\]

where \( T \) is the European basket option maturity and \( T_1 \) is the domestic and foreign zero-coupon bond maturity. The details of demonstrations of these SDE are given in Lamberton and Lapeyre (1997), and in Baxter and Rennie (1996).

We note that, under the risk neutral measure \( Q \), \( \delta_T \) follows a normal distribution. However, \( S_T, C_T, P(T, T_1) \) and \( K(T, T_1) \) are lognormally distributed, which means that \( \ln(S_T/S_t), \ln(C_T/C_t), \ln(P(T, T_1)) \) and \( \ln(K(T, T_1)) \) follow a multivariate normal distribution.

### 3 European basket option valuation

A basket option is an option on a basket or a portfolio of assets which gives its holder the right, but not the obligation, to purchase or to sell a pre-specified fixed portfolio at a fixed strike price and at a fixed maturity. The basket option allows us to cover financial risks (commodity-price, exchange-rate and interest-rate risks) through a single hedge and at a total cost lower than it would take to hedge each of these risks separately. In fact, the premium advantage of a basket option increases significantly when the basket is appropriately diversified to include negatively correlated assets. However, basket options are less liquid than standard options, as they are traded on over-the-counter markets. Once set up, they also tend to be somewhat inflexible, making them difficult to re-adjust without incurring extra expenses. Finally, basket options are more difficult to evaluate than standard options.

In the following, we suppose that the firm wants to hedge its financial risks, commodity-price fluctuations, exchange-rate risks and interest-rate risks. We thus consider a European call-basket option which gives its holder the possibility of buying, at exercise price \( K_B \), a portfolio consisting of a commodity, a domestic zero-coupon bond and a foreign zero-coupon bond expressed in local currency. We suppose that \( w_1, w_2 \) and \( w_3 \) represent, respectively, the proportions of the basket invested in the commodity, domestic zero-coupon bond and foreign zero-coupon bond converted in domestic currency.
This basket option is defined by $F_t$-measurable random variable $X_B \geq 0$, where $X_B$ is the final payment at the maturity $T$ of the basket option. We assume a European call option, but it is important to note that the same reasoning can apply to a European put option. Though

$$X_B = \text{Max} [w_1 S_T + w_2 P(T,T_1) + w_3 Y(T,T_1) - K_B, 0],$$

such as $Y(T,T_1) = K(T,T_1) \times C_T$.

Since the market model that we developed in the previous section is complete, the price of an accessible contingent claim is equal to the expectation, under the martingale measure $Q$, of the assets’ future values, discounted at the risk-free interest rate (Harisson and Pliska (1981)). Consequently, the basket-call-option price is given by

$$V_t^{basket} = E_Q \left[ \exp \left( - \int_t^T r_u du \right) X_B \mid \mathcal{F}_t \right] = P(t,T) E_{Q_T} \left[ X_B \mid \mathcal{F}_t \right] = P(t,T) \int_{\mathbb{R}^3} \text{Max} (w_1 s + w_2 p + w_3 y - K_B, 0) \times f(s,p,y) dsdpdy,$$

where $f(s,p,y)$ is the joint distribution of the risky assets $S_T$, $P(T,T_1)$, $C_T K(T,T_1)$ under the T-forward measure $Q_T$, the details of the computation of the $Q_T$ measure are given in Appendix B. This new martingale measure $Q_T$ has Radon-Nikodym derivative with respect to $Q$ of

$$\frac{dQ_T}{dQ} = \frac{\exp \left( - \int_t^T r_u du \right)}{P(t,T)}.$$

When deriving the joint density function of risky assets $f(s,p,y)$ in Appendix C, we found

$$f(s,p,y) = \frac{1}{spvy \sqrt{2\pi} \left\| \Sigma_{Q_T} \right\|} \exp \left\{ -\frac{1}{2} \left( x - \mu_{Q_T} \right)^T \Sigma_{Q_T} (x - \mu_{Q_T}) \right\},$$

where $x_T$ is a multi-normal Gaussian vector of the realized returns of risky assets under the T-forward measure $Q_T$. In fact

$$x = \begin{pmatrix} \ln \left( \frac{S_T}{\pi} \right) \\ \ln (P(T,T_1)) \\ \ln (Y(T,T_1)) \end{pmatrix}.$$

$\mu_{Q_T}$ represents the vector of expected returns on risky assets under the T-forward measure $Q_T$ and is given in equation (5),

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Finally, more techniques, such as multinomial trees or Quasi-Monte Carlo Methods, can be used to price multiple-proposed. We can use either analytical approximations or numerical techniques. Several numerical integral is not available. In the literature, various methods for valuing basket options have been proposed. We use Monte Carlo simulations to solve equation (4) numerically. To do so, we follow this procedure:

\[ \mu_{Q_T} = \left( \begin{array}{c} \int_t^T f(0, u) du - (m_1 + \frac{1}{2} \sigma_S^2)(T - t) + (m_1 - \delta_t) \left( \frac{1-e^{-a(T-t)}}{a} \right) \\ + \frac{1}{6} \sigma_f^2 (T^3 - t^3) - \frac{1}{2} \sigma_S \sigma_f \rho_{13} (T - t)^2 \\ - \int_T^T f(0, u) du - \frac{1}{6} \sigma_f^2 (T_1 - T)^3 \\ \int_t^T f(0, u) du - \int_{T_1}^T f(0, u) du - \frac{1}{2} \sigma_S^2 (T - t) + \frac{1}{2} \sigma_S \sigma_C (T - t) \\ + \frac{1}{6} \left( \sigma_f^2 - \sigma_g^2 \right) (T^3 - t^3) + (\sigma_g \sigma_C T - \frac{1}{2} \sigma_g^2 TT_1) (T_1 - T) \\ - \frac{1}{2} \sigma_C \sigma_f \rho_{23} (T - t)^2 - \frac{1}{2} \sigma_g \sigma_f \rho_{34} (T_1 - T)^2 + \ln (C_t) \end{array} \right). \]

Finally, \( \Sigma_{Q_T} \) represents the variance-covariance matrix of risky-asset returns under the T-forward measure \( Q_T \) which is given by the equation (21) in Appendix C. The different elements of this matrix are defined below:

\[
\begin{align*}
\text{Var1} & = \left[ \frac{\sigma_f^2 + \sigma_g^2 + \sigma^2_C}{2\sigma^2} \right] (T - t) \\
& \quad + \left[ \frac{1}{2\sigma^2} \left( 1-e^{-2a(T-t)} \right) \right] \left( \sigma_S \delta + \sigma_f \delta \rho_{13} - \frac{\sigma_f^2}{a^2} \right) \left( \frac{1-e^{-a(T-t)}}{a} \right), \\
\text{Var2} & = \sigma_f^2 (T_1 - T)^2 T, \\
\text{Var3} & = \left[ \left( \frac{\sigma_f^2 + \sigma_g^2 + \sigma^2_C}{2\sigma^2} \right) + 2\sigma_f \sigma_C \rho_{23} + 2\sigma_g^2 (T_1 - T) \\
& \quad - (2\sigma_g \sigma_C \rho_{24} + 2\sigma_f \sigma_g \rho_{34}) (1 + (T_1 - T)) \right] (T - t) + \sigma_g^2 T (T_1 - T)^2, \\
\text{Cov12} & = \sigma_f (T_1 - T) \left[ - \left( \sigma_f + \rho_{13} \left( \sigma_S - \frac{\sigma_S \delta}{a} \right) \right) (T - t) - \rho_{13} (T - t) \right] \left( \frac{1-e^{-a(T-t)}}{a} \right), \\
\text{Cov13} & = \left[ \sigma_f^2 + \sigma_f \sigma_C \rho_{23} - \sigma_f \sigma_g \rho_{34} (1 + T_1 - T) \\
& \quad + (\sigma_f \rho_{13} - \sigma_g \rho_{14} (1 + T_1 - T) + \sigma_C \rho_{12}) \left( \sigma_S - \frac{\sigma_S \delta}{a} \right) \right] (T - t), \\
\text{Cov23} & = \sigma_f \sigma_g T (T_1 - T)^2 \rho_{34} + \sigma_f (T_1 - T) (T - t) \left[ - \sigma_f + \sigma_g \rho_{34} - \sigma_C \rho_{23} \right].
\end{align*}
\]

Thus, to value the European basket option given by equation (4), we have to solve a multidimensional integral. Unfortunately, it appears that the closed-form solution to this multidimensional integral is not available. In the literature, various methods for valuing basket options have been proposed. We can use either analytical approximations or numerical techniques. Several numerical techniques, such as multinomial trees or Quasi-Monte Carlo Methods, can be used to price multiple-asset contingent claims. For the specific purposes of this paper which are to design a basket-option model and estimate its parameters, we will apply simple Monte Carlo method, as it is known to be more efficient at valuing multidimensional European contingent claims.

### 3.1 Monte Carlo Method

We use Monte Carlo simulations to solve equation (4) numerically. To do so, we follow this procedure:

**Step 1:** We generate a vector \( Z \) of standard normal random variables independent and identically distributed. This vector is defined by
\[ Z = \left( Z^{(1)}, Z^{(2)}, Z^{(3)}, Z^{(4)} \right)^{'} , \]

**Step 2:** Because our model uses correlated normal random variables, we have to define a multivariate normal-random vector \( X = \left( X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)} \right)^{'} \) such that,

\[ X = CZ , \]

where \( C \) is a \( 4 \times 4 \) matrix satisfying \( CC' = \Gamma Q_T \) and \( \Gamma Q_T \) represents the correlation matrix between the Brownian motions. The values of risky assets at time \( T \) under the \( T \)-forward measure \( Q_T \) are thus given by,

\[
S_T = S_0 \exp \left[ \int_0^T f(0, u)du - (m_1 + \frac{1}{2}\sigma_S^2)T + (m_1 - \delta_0) \left( \frac{1-e^{-aT}}{a} \right) + \frac{1}{6}\sigma_f^2 T^3 - \frac{1}{2}\sigma_S \sigma_f \rho_{13} T^2 + \sigma_f \sqrt{T} X^{(3)} \right] ,
\]

\[
P(T, T_1) = \exp \left[ - \int_T^{T_1} f(0, u)du - \frac{1}{6}\sigma_f^2 (T_1 - T)^3 - \sigma_f (T_1 - T) \sqrt{T} X^{(3)} \right] ,
\]

\[
Y(T, T_1) = C_0 \exp \left[ \int_0^T f(0, s)ds - \int_0^{T_1} g(0, u)du + \frac{1}{6} \left( \sigma_f^2 - \sigma_f^2 \right) T^3 - \frac{1}{2}\sigma_C^2 T + \frac{1}{2}\sigma_f \sigma_C T^2 - \frac{1}{2}\sigma_f \sigma_T \rho_{34} (T_1 - T)^2 \right]
\]

where \( \equiv \) means equality in distribution.

**Step 3:** We compute the basket option value for the trajectory \( i \). Indeed,

\[
V_{i,0}^{\text{basket}} = P(0, T) \max \left( w_1 S_{i,T} + w_2 P_{i,T_1} + w_3 Y_{i,(T,T_1)} - K_B, 0 \right) .
\]

**Step 4:** The first three steps must be repeated \( N \) times to obtain \( N \) option values. The Monte-Carlo estimator of the basket option value is equal to,

\[
\hat{V}_0^{\text{basket}} = \frac{1}{N} \sum_{i=1}^{N} V_{i,0}^{\text{basket}} .
\]
3.2 Numerical results

The results of the basket-option valuation are presented in Table 1. The parameters were fixed arbitrarily. Standard deviations of the estimates are calculated so as to assess the precision of the Monte-Carlo estimator of the basket-option value. The numerical solutions obtained show that the prices converge rapidly. However, results depend on the exercise price of the basket option. Simulation results are precise at 0.01, as derived from 100 000 trajectories for an in-the-money basket option, from 50 000 trajectories for an at-the-money basket option and from 5 000 trajectories for an out-of-the-money basket option. We note that the value of the basket varies when we change the proportion of the assets composing the basket.

Table 1

<table>
<thead>
<tr>
<th>Sample size</th>
<th>In-the-money</th>
<th>At-the-money</th>
<th>Out-of-the-money</th>
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<tbody>
<tr>
<td></td>
<td>$K_B = 0.8 \times V$</td>
<td>$K_B = V$</td>
<td>$K_B = 1.2 \times V$</td>
</tr>
<tr>
<td>MC price</td>
<td>Std</td>
<td>MC price</td>
<td>Std</td>
</tr>
<tr>
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<td>5.4125</td>
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<td>1.0370</td>
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<tr>
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<td>0.1422</td>
<td>1.0870</td>
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<tr>
<td>300 000</td>
<td>5.3961</td>
<td>0.0062</td>
<td>1.2169</td>
</tr>
</tbody>
</table>

MC price is the basket option value given by the Monte-Carlo simulations and Std is the standard deviation of the Monte-Carlo estimator. $S_0 = 100$, $C_0 = 1.6$, $P(0, T) = 0.93$, $\delta_0 = 0.2$, $r_0 = 0.05$, $u_0 = 0.035$, $\sigma_S = 0.2$, $\sigma_C = 0.15$, $\sigma_\delta = 0.35$, $\sigma_f = 0.04$, $\sigma_g = 0.05$, $\alpha_S = 0.3$, $\kappa = 1.8$, $\theta = 0.18$, $w_1 = 0.3$, $w_2 = 0.4$ and $w_3 = 0.3$, $T = 126$, $T_1 = 189$, $\Delta_t = \frac{1}{50}$

4 Parameters estimation

In this section, we propose to estimate the various parameters of our valuation model. Since the basket-option valuation model presented in Section 2 supposes normality of returns on risky assets, the maximum likelihood method appears to be the relevant one for estimating our whole set of parameters. This estimation technique can, in fact, incorporate all the information related to the
distribution of the random variables. What recommends the maximum likelihood approach is its capacity to estimate several unknown parameters simultaneously. Maximum likelihood estimators are also attractive because of their asymptotic properties. They are both consistent, since they converge to the exact value of an increasing sample size, and asymptotic efficient, since they show the least variance compared with all other unbiased estimators.

The maximum likelihood method consists in estimating the unknown parameters which would maximize the probability of obtaining the sample that one has actually observed. In what follows, we shall show how the maximum likelihood method can be used to estimate the parameters of the basket-option valuation model suggested in Section 2.

4.1 Global likelihood function

We shall follow the methodology described below to derive the global likelihood function for the commodity price, the exchange rate as well as the domestic and foreign zero-coupon bonds. Calculating this function in Appendix E, we obtain

$$L (J; \mu_p, \Sigma_P) = \prod_{i=2}^{N} \frac{1}{s_{t_i} c_{t_i} f_{t_i}^D f_{t_i}^F} \times \left( \frac{1}{\sqrt{2\pi}} \right)^{N-1} \times \left( \frac{1}{\sqrt{\|\Sigma_P\|}} \right)^{N-1} \times \prod_{i=2}^{N} \exp \left[ -\frac{1}{2} (x_{t_i} - \mu_P)^\top \Sigma_P^{-1} (x_{t_i} - \mu_P) \right],$$

where $J = \{J_{t_i} = (s_{t_i}, c_{t_i}, f_{t_i}^D f_{t_i}^F) : i = (1, 2, ...N)\}$ is the vector of risky assets, $\mu_P$ represents the expected returns on risky assets given by equation (24) in Appendix E, and $\Sigma_P$ is the variance-covariance matrix of risky-asset returns in equation (25) of Appendix E, and, finally, $x_{t_i}$ represents the realized returns on risky assets,

$$x_{t_i} = \begin{pmatrix} \ln \left( \frac{s_{t_i}}{s_{t_{i-1}}} \right) \\ \ln \left( \frac{c_{t_i}}{c_{t_{i-1}}} \right) \\ \ln \left( \frac{f_{t_i}^D}{f_{t_{i-1}}^D} \right) \\ \ln \left( \frac{f_{t_i}^F}{f_{t_{i-1}}^F} \right) \end{pmatrix},$$

where $(s_{t_i}, c_{t_i}, f_{t_i}^D, f_{t_i}^F)'$ is the realization of the random vector $(S_{t_i}, C_{t_i}, F_D(t_i, T_1, T_2), F_F(t_i, T_1, T_2))'$. $F_D(t, T_1, T_2)$ and $F_F(t, T_1, T_2)$ are the prices at time $t$ of futures contracts written on the domestic and foreign zero-coupon bonds respectively. $T_1$ is the maturity of futures contracts. We suppose that the domestic and foreign zero-coupon bonds have the same maturity $T_2$.

We have to find the values of $\mu_P$ and $\Sigma_P$ that maximize the likelihood function described by equation (7). Because the maximum of $L (J; \mu_p, \Sigma_P)$ is reached at the same point as the maximum of $\ln \left( L (J; \mu_p, \Sigma_P) \right)$, $\mu_P^\wedge$ and $\Sigma_P^\wedge$ are the maximization’s solutions of this global log-likelihood function:
\[ L = - \sum_{i=2}^{N} \left[ \ln(s_{t_i}) + \ln(c_{t_i}) + \ln(f_{t_i}^D) + \ln(f_{t_i}^F) \right] \\
- \frac{N-1}{2} \ln \| \Sigma_P \| - \frac{1}{2} \sum_{i=2}^{N} [(x_{t_i} - \mu_P)' \Sigma_P^{-1} (x_{t_i} - \mu_P)]. \] (8)

4.2 Estimation procedure

Although direct optimization of the log-likelihood function in equation (8) may appear simple, it is actually not so in practice. Because a large number of parameters must be estimated together, we shall avoid convergence problems by proceeding in three steps. We shall first consider the parameters relating to the domestic and foreign zero-coupon bonds; then, supposing that these parameters are known, we estimate the ones related to the commodity price and the exchange rate as well as to the different elements in the correlation matrix. Finally, like Duan (1994), we estimate the commodity convenience yield and the parameters which characterize it.

4.2.1 Estimating the parameters of the zero-coupon bonds

Since we assume that the domestic and foreign forward rates will follow the HJM model, their parameters cannot be directly estimated. Under the objective probability measure, the HJM approach requires specification of the initial term structure, the volatility associated with forward rates, and the market price of risk. Basically, the HJM model’s estimation problems arise from two facts: (i) the computational complexity of carrying out the term structure’s entire history to account for the usually path-dependent nature of spot rates and (ii) the difficulty of working with the market price of risk. The estimation approach most often adopted consists in reducing the system to a Markovian form under some particular functional specification of forward-rate volatility. Chiarella and Kwon (2001), Bhar and Chiarella (1997), and Ritchken and Sankarasubramanian (1995) provide theories on reduction to Markovian form. However, in this paper, we choose HJM’s constant instantaneous forward-rate volatility for its Markovian property. The problem now remaining is how to handle the market price of risk. As in Bhar, Chiarella, and To (2001) (BCT hereafter), we use the analytical formulae for futures prices on zero-coupon bonds, thus avoiding the need to consider the market price of risk in our estimation (see Appendix D for details). Unfortunately, these analytical results apply only to a limited number of volatility specifications such as the constant volatility one used in this paper.

As futures contracts on zero-coupon bonds are lognormally distributed, we can easily derive their multivariate likelihood function given by:
where \( \eta_p \) represents the expected futures' returns on zero-coupon bonds under the probability measure \( P \),

\[
\eta_p = \left( -\frac{1}{2\sigma_f^2} (T_2 - T_1)^2 \Delta_t \right) \left( -\frac{1}{2\sigma_g^2} (T_2 - T_1)^2 \Delta_t \right),
\]

\( \Phi_P \) is the variance-covariance matrix of futures' returns on zero-coupon bonds under the probability measure \( P \),

\[
\Phi_P = \begin{pmatrix}
\sigma_f^2 (T_2 - T_1)^2 \Delta_t & \sigma_f \sigma_g (T_2 - T_1)^2 \rho_{34} \Delta_t \\
\sigma_f \sigma_g (T_2 - T_1)^2 \rho_{34} \Delta_t & \sigma_g^2 (T_2 - T_1)^2 \Delta_t
\end{pmatrix},
\]

\( \rho_{34} \) is the correlation coefficient between the domestic and foreign zero-coupon bond.

\( x_{t_i} \) represents the realized returns of futures on zero-coupon bonds,

\[
x_{t_i} = \begin{pmatrix}
\ln \left( \frac{f^D_{t_i}}{f^F_{t_i}} \right) \\
\ln \left( \frac{f^F_{t_i}}{f^D_{t_i}} \right)
\end{pmatrix},
\]

where \( (f^D_{t_i}, f^F_{t_i})' \) is the realization of the random vector \( (F_D (t_i, T_1, T_2), F_F (t_i, T_1, T_2))' \).

We have to find the values of \( \sigma_f, \sigma_g \) and \( \rho_{34} \) that maximize the likelihood function in equation (9). Because of the monotonicity of the logarithmic transformation, the maximum of

\[
L \left( F_D (t_i, T_1, T_2), F_F (t_i, T_1, T_2), i = 1, 2, \ldots, N; \mu_P, \Sigma_P \right)
\]

is the same as the maximum of

\[
\ln \left( L \left( F_D (t_i, T_1, T_2), F_F (t_i, T_1, T_2), i = 1, 2, \ldots, N; \mu_P, \Sigma_P \right) \right)
\]

Thus, \( \hat{\sigma}_f, \hat{\sigma}_g \) and \( \hat{\rho}_{34} \) are the solutions of the maximization of the function \( L \) below:

\[
L = -\sum_{i=2}^{N} \left[ \ln \left( f^D_{t_i} \right) + \ln \left( f^F_{t_i} \right) \right] - \frac{N - 1}{2} \ln \| \Phi_P \| - \frac{1}{2} \sum_{i=2}^{N} \left[ (x_{t_i} - \eta_P)' \Phi_P^{-1} (x_{t_i} - \eta_P) \right].
\]

Using the estimated parameters related to the domestic and foreign zero-coupon bonds, we can now estimate the parameters \( \hat{\alpha}_S, \hat{\alpha}_C, \hat{\sigma}_S, \hat{\sigma}_C, \hat{\rho}_{12}, \hat{\rho}_{13}, \hat{\rho}_{14}, \hat{\rho}_{23}, \hat{\rho}_{24} \) and \( \hat{\rho}_{34} \) that maximize the global likelihood function (8).
4.2.2 Estimation of the convenience yield and its parameters

Given that the convenience yield is not observable, estimating its parameters is rather complicated. Nevertheless, this is what we attempt to do in this section. Schwartz (1997) proposes to use Kalman filters to estimate the parameters of a similar model as well as the time series of the unobservable state variables. His methodology consists in first, putting the model in state space form to be able thereafter, to apply the Kalman filters. This technique is a recursive procedure capable of computing the optimal estimator of the state variable at time \( t \), based on the information available at that time, and allowing the estimate of the state vector to be continuously updated as new information becomes available. It is a two-step procedure. The first step consists in using currently available information to make an optimal forecast of the next observation; the second step incorporates the new observation into the estimator of the state variable.

Since Kalman filters are based on the discretization of the stochastic processes, we think that this introduces a bias, so we propose to use the maximum likelihood method to estimate the time series of the convenience yield as well as the parameters describing its behavior.

The basic idea of using the likelihood function for unobservable variables is taken from Duan (1994). This technique consists in finding a functional relationship between the observable variables and the unobservable variables. Thereafter, by using the chronological data on the observable variables, we can call on to the transformation data method to obtain the maximum likelihood estimates of the unobservable variables. We shall follow this methodology to estimate the time series of the convenience yield as well as its parameters.

Deriving the value of a commodity futures contract maturing at time \( T_3 \) in Appendix F, we found:

\[
F(t, T_3) = S_t \exp \left\{ \int_t^{T_3} f(0, u) du + \frac{1}{2} \sigma_f^2 (T_3^2 - t^2) + \frac{\sigma_f^2}{a} \left( \frac{1-e^{-2a(T_3-t)}}{a} \right) \right. \\
+ \left[ -m + \frac{\sigma_f^2}{2a} - \frac{\sigma_f^2}{a} + \frac{1}{2} \sigma_f^2 + \rho_{13} \sigma_f (\sigma - \frac{\sigma}{a}) \right] (T_3 - t) \\
+ \left[ \frac{\sigma_f}{a} (\sigma - \frac{\sigma}{a} + \rho_{13} \frac{\sigma_f}{a}) + (m - \delta_t) \right] \left( \frac{1-e^{-a(T_3-t)}}{a} \right) \right\}. \tag{11}
\]

Remind that the future contract price and the commodity price are observable, the initial structure of interest rates and the future contract maturity are known and finally the parameters of the commodity price as well as the domestic zero-coupon bond have already been estimated by the maximum likelihood method in the previous subsection. Consequently, equation (11) depends only on the convenience yield and its parameters \( \kappa, \theta \) and \( \sigma_\delta \). We deduce a functional relationship between the price of the commodity future contract which is observable and the convenience yield which is unobservable. Thus, we can use the transformation data method to find the likelihood function of the commodity future contract. The resulting likelihood function becomes the likelihood function of the implied convenience yield multiplied by the Jacobian of the transformation evaluated at the implied convenience yield. (see Appendix G)
\[
L(F(t_i, T_3) : i \in (1, 2, ..., N) ; \kappa, \theta, \sigma_\delta) = \left( \frac{1}{\sqrt{2\pi}} \right)^{N-1} \times \left( \frac{\sqrt{2\kappa}}{\sigma_\delta \sqrt{1 - e^{-2\kappa \Delta t}}} \right)^{N-1} \\
\times \prod_{i=2}^{N} \exp \left[ -\frac{\kappa \left( \hat{\delta}_{t_i} - \delta_{t_{i-1}} e^{-\kappa \Delta t} - \theta (1 - e^{-\kappa \Delta t}) \right)^2}{\sigma_\delta^2 (1 - e^{-2\kappa \Delta t})} \right] \times \prod_{i=1}^{N} \left| \frac{-a}{(1 - e^{-a(T_2 - t_i)}) F(t_i, T_3)} \right|
\]

where \( \hat{\delta}_{t_i} \) is given by the equation (26) of Appendix G,

\[
\hat{\delta}_{t_i} = \left( \frac{a}{1 - e^{-a(T_3 - t_i)}} \right) \left\{ \ln \left( \frac{S_{t_i}}{F(t_i, T_3)} \right) + \int_{t_i}^{T_3} f(u)du + \frac{\sigma_\delta^2}{\kappa} f(T_3 - t_i) \right\} + m \left[ -m + \frac{\sigma_\delta^2}{2a^2} - \frac{\sigma_\delta \sigma_{\delta f}}{a} + \frac{1}{2} \sigma_{\delta f}^2 + \rho_{13} \sigma_f \left( \sigma_s - \frac{\sigma_\delta}{a} \right) \right] (T_3 - t_i) \]

\[
+ m + \frac{\sigma_\delta}{a} \left( \sigma_s - \frac{\sigma_\delta}{a} + \rho_{13} \frac{\sigma_f}{a} \right).
\]

We have to find the values of \( \kappa, \theta \) and \( \sigma_\delta \) that maximize the above likelihood function, given our observations. Since the maximum of

\[
\ln(L(F(t_i, T_3) : i \in (1, 2, ..., N) ; \kappa, \theta, \sigma_\delta))
\]

is the same as the maximum of \( L(F(t_i, T_3) : i \in (1, 2, ..., N) ; \kappa, \theta, \sigma_\delta) \), \( \hat{\kappa}, \hat{\theta} \) and \( \hat{\sigma}_\delta \) are the maximization solutions of the following log-likelihood function \( L \):

\[
L = \left( \frac{N - 1}{2} \right) \ln(2\kappa) - (N - 1) \ln(\sigma_\delta) - \left( \frac{N - 1}{2} \right) \ln(1 - e^{-2\kappa \Delta t})
\]

\[
- \frac{\kappa}{\sigma_\delta^2 (1 - e^{-2\kappa \Delta t})} \sum_{i=2}^{N} \left( \hat{\delta}_{t_i} - \delta_{t_{i-1}} e^{-\kappa \Delta t} - \theta (1 - e^{-\kappa \Delta t}) \right)^2
\]

\[
+ \sum_{i=1}^{N} \ln \left| \frac{-a}{(1 - e^{-a(T_3 - t_i)}) F(t_i, T_3)} \right|.
\]

(12)

To maximize numerically the three likelihood functions given by equations (8), (10) and (12), we use the quadratic hill-climbing algorithm of Goldfeld, Quandt and Trotter (1996) with a convergence criterion based on the absolute values of the variations in parameter values and functional values between successive iterations. When both of these changes are smaller than 1e-5, we attend convergence. It is clear that the three-step estimation procedure described above employs many simplifications. We now use a Monte-Carlo study to evaluate its performance.
4.3 Monte Carlo study

In this subsection, we conduct a Monte Carlo study to evaluate the quality of the coefficients estimated using the maximum likelihood method. We apply simulation techniques to verify whether the normal distribution given in subsection 4.2 adequately approximate the parameter estimates for a sample size $N$. In other words, we assess how well the normal distribution proposed by the theory approximates the empirical distribution for a reasonable sample size. Similarly, we check the distributions for domestic and foreign zero-coupon bonds, global model and convenience yield. We simulate the data on a daily basis. For each simulated data, we conduct the maximum likelihood estimation and compute the point estimates and the associated variance. We repeat the simulation run 2000 times to obtain the Monte Carlo estimate for the relevant quantities. This procedure is described below:

1. **Check of the domestic and foreign zero-coupon bonds’ distribution**: we simulate the data $F_D = \{F_D(t_i, T_1, T_2) : i \in \{1, 2, ..., 250\}\}$ and $F_F = \{F_F(t_i, T_1, T_2) : i \in \{1, 2, ..., 250\}\}$ such that

   \[
   F_D(t_i, T_1, T_2) \equiv F_D(t_{i-1}, T_1, T_2) \exp \left[ -\frac{1}{2} \sigma_f^2 (T_2 - T_1)^2 \Delta_t - \sigma_f(T_2 - T_1) \sqrt{\Delta_t} X_{t_i}^{(3)} \right],
   \]

   \[
   F_F(t_i, T_1, T_2) \equiv F_F(t_{i-1}, T_1, T_2) \exp \left[ -\frac{1}{2} \sigma_g^2 (T_2 - T_1)^2 \Delta_t - \sigma_g(T_2 - T_1) \sqrt{\Delta_t} X_{t_i}^{(4)} \right],
   \]

   where $X_{t_i} = \left( X_{t_i}^{(3)} \quad X_{t_i}^{(4)} \right)'$ is a vector of correlated standard normal random variables. The corresponding correlation coefficient is equal to $\rho_{34}$. Table 2 presents the specific parameters values used in the simulation.

<table>
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<tr>
<th></th>
<th>$\hat{\sigma}_f$</th>
<th>$\hat{\sigma}_g$</th>
<th>$\hat{\rho}_{34}$</th>
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<td>0.0500</td>
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<td>0.7560</td>
<td>0.7425</td>
</tr>
<tr>
<td>90 % cvr</td>
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<td>95 % cvr</td>
<td>0.9445</td>
<td>0.9465</td>
<td>0.9450</td>
</tr>
<tr>
<td>99 % cvr</td>
<td>0.9850</td>
<td>0.9865</td>
<td>0.9880</td>
</tr>
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</table>
True is the parameter value used in the Monte Carlo simulation; Mean, Median and Std are the sample statistics computed with the 2000 estimated parameter values; cvr is the coverage rate defined as the percentage of the 2000 parameter estimates for which the true parameter value is contained in the $\alpha$ confidence interval implied by the asymptotic distribution; $F_D^0 = 0.93$, $F_F^0 = 0.95$, $T_1 = 250$, $T_2 = 280$, and $N = 250$

2. **Check of the global model’s distribution:** we simulate the following risky assets trajectories: $S = \{S_{t_i} : i \in \{1, 2, ..., 250\}\}$, $C = \{C_{t_i} : i \in \{1, 2, ..., 250\}\}$,

$F_D = \{F_D(t_i, T_1, T_2) : i \in \{1, 2, ..., 250\}\}$ and $F_F = \{F_F(t_i, T_1, T_2) : i \in \{1, 2, ..., 250\}\}$ such that,

\[
S_{t_i} \overset{i.i.d.}{=} S_{t_{i-1}} \exp \left[ \left( \alpha_S - \frac{1}{2} \sigma_S^2 \right) \Delta t + \sigma_S \sqrt{\Delta t} X_{t_i}^{(1)} \right],
\]

\[
C_{t_i} \overset{i.i.d.}{=} C_{t_{i-1}} \exp \left[ \left( \alpha_C - \frac{1}{2} \sigma_C^2 \right) \Delta t + \sigma_C \sqrt{\Delta t} X_{t_i}^{(2)} \right],
\]

\[
F_D(t_i, T_1, T_2) \overset{i.i.d.}{=} F_D(t_{i-1}, T_1, T_2) \exp \left[ -\frac{1}{2} \sigma_f^2 (T_2 - T_1)^2 \Delta t - \sigma_f (T_2 - T_1) \sqrt{\Delta t} X_{t_i}^{(3)} \right],
\]

\[
F_F(t_i, T_1, T_2) \overset{i.i.d.}{=} F_F(t_{i-1}, T_1, T_2) \exp \left[ -\frac{1}{2} \sigma_g^2 (T_2 - T_1)^2 \Delta t - \sigma_g (T_2 - T_1) \sqrt{\Delta t} X_{t_i}^{(4)} \right],
\]

where $X_{t_i} = \begin{pmatrix} X_{t_i}^{(1)} & X_{t_i}^{(2)} & X_{t_i}^{(3)} & X_{t_i}^{(4)} \end{pmatrix}$ is the vector of correlated standard normal random variables. Table 3 presents the specific parameters values used in the simulation.
Table 3
Simulation results for the global likelihood function

<table>
<thead>
<tr>
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<th>$\hat{\alpha}_C$</th>
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<th>$\hat{\sigma}_C$</th>
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</tbody>
</table>

25 % cvr 0.2615 0.2525 0.2740 0.2585 0.2630 0.2535 0.2650 0.2315 0.2630 0.2395
50 % cvr 0.4995 0.4850 0.5195 0.5075 0.5065 0.4805 0.5195 0.5005 0.5010 0.4875
75 % cvr 0.7390 0.7320 0.7500 0.7510 0.7345 0.7415 0.7430 0.7600 0.7535 0.7415
90 % cvr 0.8935 0.8935 0.8990 0.9065 0.8895 0.8935 0.8985 0.8980 0.8970 0.8965
95 % cvr 0.9325 0.9515 0.9450 0.9515 0.9325 0.9425 0.9500 0.9440 0.9465 0.9425
99 % cvr 0.9680 0.9865 0.9895 0.9875 0.9830 0.9805 0.9890 0.9840 0.9870 0.9840

True is the parameter value used in the Monte Carlo simulation; Mean, Median and Std are the sample statistics computed with the 2000 estimated parameter values; cvr is the coverage rate defined as the percentage of the 2000 parameter estimates for which the true parameter value is contained in the $\alpha$ confidence interval implied by the asymptotic distribution; $S_0 = 100$, $C_0 = 1.6$, $F_D^0 = 0.93$, $F_F^0 = 0.95$, $T_1 = 250$, $T_2 = 280$, and $N = 250$
3. **Check of the convenience yield distribution**: we simulate $S = \{S_{t_i} : i \in \{1, 2, ..., 250\}\}$ and $\delta = \{\delta_{t_i} : i \in \{1, 2, ..., 250\}\}$ such that,

$$S_{t_i} \equiv S_{t_{i-1}} \exp \left[ \left( \alpha_S - \frac{1}{2} \sigma_S^2 \right) \Delta_t + \sigma_S \sqrt{\Delta_t} Z_{t_i}^{(1)} \right],$$

$$\delta_{t_i} \equiv \delta_{t_{i-1}} e^{-\kappa \Delta_t} + \theta \left( 1 - e^{-\kappa \Delta_t} \right) + \frac{\sigma_{\delta}}{\sqrt{2\kappa}} \sqrt{1 - e^{-2\kappa \Delta_t}} Z_{t_i}^{(1)},$$

where $Z_{t_i}^{(1)}$ is a vector of a standard normal random variables. Assuming that the future contract maturity is equal to 252 days and the initial structure of interest rates is flat, we deduce $F = \{F(t_i, T_3) : i \in \{1, 2, ..., 252\}\}$ using equation (11). We can use the maximum likelihood estimates of $\hat{\kappa}$, $\hat{\theta}$ and $\hat{\sigma}_\delta$ to evaluate the implicit convenience yield $\hat{\delta}$ given by equation (26) of Appendix H. We evaluate the estimation bias at each replication since the convenience yield value changes at each repetition of the procedure.

**Table 4**  
Simulation results for the convenience yield’s likelihood function

<table>
<thead>
<tr>
<th></th>
<th>$\hat{\kappa}$</th>
<th>$\hat{\theta}$</th>
<th>$\hat{\sigma}_\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>True</strong></td>
<td>1.8000</td>
<td>0.1800</td>
<td>0.3500</td>
</tr>
<tr>
<td><strong>Mean</strong></td>
<td>1.9144</td>
<td>0.1741</td>
<td>0.3565</td>
</tr>
<tr>
<td><strong>Median</strong></td>
<td>1.8821</td>
<td>0.1821</td>
<td>0.3541</td>
</tr>
<tr>
<td><strong>Std</strong></td>
<td>0.4882</td>
<td>0.2303</td>
<td>0.0417</td>
</tr>
<tr>
<td><strong>25 % cvr</strong></td>
<td>0.2440</td>
<td>0.2365</td>
<td>0.2525</td>
</tr>
<tr>
<td><strong>50 % cvr</strong></td>
<td>0.4615</td>
<td>0.4510</td>
<td>0.4850</td>
</tr>
<tr>
<td><strong>75 % cvr</strong></td>
<td>0.7180</td>
<td>0.6810</td>
<td>0.7475</td>
</tr>
<tr>
<td><strong>90 % cvr</strong></td>
<td>0.8945</td>
<td>0.8380</td>
<td>0.8955</td>
</tr>
<tr>
<td><strong>95 % cvr</strong></td>
<td>0.9440</td>
<td>0.8890</td>
<td>0.9525</td>
</tr>
<tr>
<td><strong>99 % cvr</strong></td>
<td>0.9805</td>
<td>0.9285</td>
<td>0.9855</td>
</tr>
</tbody>
</table>

True is the parameter value used in the Monte Carlo simulation; Mean, Median and Std are the sample statistics computed with the 2000 estimated parameter values; cvr is the coverage rate defined as the percentage of the 2000 parameter estimates for which the true parameter value is contained in the $\alpha$ confidence interval implied by the asymptotic distribution; $S_0 = 100$, $\delta_0 = 0.2$, $T_3 = 252$, $r_0 = 0.05$ and $N = 250$

Tables 2, 3 and 4 present the simulations results for the zero-coupon bonds’ likelihood function, global likelihood function and the convenience yield likelihood function respectively. The simulation results reveal that, for all the parameters, the maximum likelihood estimator is unbiased. The mean and median of the estimates are very close to their true value and the standard deviations of the
estimates are relatively small. The coverage rates\(^2\) indicate that the normal distribution is a good approximation of the small sample distribution.

However, concerning the parameters related to the convenience yield, we note that the precision of the estimates of the speed of adjustment coefficient \(\hat{\kappa}\) and the long run mean \(\hat{\theta}\) is not satisfactory. The standard deviations of the estimates are 0.4882 and 0.2303 respectively, this is probably due to the small sample size, only 250 observations. To check this hypothesis, we reproduced the study using a sample of 502 observations on futures contracts. For mean-reverting processes we expect the long-run mean to converge better when we increase the sample size. The results are reported in Table 5.

### Table 5
Simulation results for the convenience yield’s likelihood function

<table>
<thead>
<tr>
<th></th>
<th>(\hat{\kappa})</th>
<th>(\hat{\theta})</th>
<th>(\hat{\sigma})</th>
</tr>
</thead>
<tbody>
<tr>
<td>True</td>
<td>1.8000</td>
<td>0.1800</td>
<td>0.3500</td>
</tr>
<tr>
<td>Mean</td>
<td>1.8625</td>
<td>0.1758</td>
<td>0.3536</td>
</tr>
<tr>
<td>Median</td>
<td>1.8544</td>
<td>0.1718</td>
<td>0.3526</td>
</tr>
<tr>
<td>Std</td>
<td>0.3292</td>
<td>0.2196</td>
<td>0.0285</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Coverage Rate</th>
<th>(\text{cvr})</th>
<th>(\text{cvr})</th>
<th>(\text{cvr})</th>
</tr>
</thead>
<tbody>
<tr>
<td>25 %</td>
<td>0.2340</td>
<td>0.2510</td>
<td>0.2530</td>
</tr>
<tr>
<td>50 %</td>
<td>0.4695</td>
<td>0.4575</td>
<td>0.4840</td>
</tr>
<tr>
<td>75 %</td>
<td>0.7265</td>
<td>0.6420</td>
<td>0.7240</td>
</tr>
<tr>
<td>90 %</td>
<td>0.8875</td>
<td>0.7855</td>
<td>0.8995</td>
</tr>
<tr>
<td>95 %</td>
<td>0.9345</td>
<td>0.8395</td>
<td>0.9460</td>
</tr>
<tr>
<td>99 %</td>
<td>0.9680</td>
<td>0.8825</td>
<td>0.9835</td>
</tr>
</tbody>
</table>

True is the parameter value used in the Monte Carlo simulation; Mean, Median and Std are the sample statistics computed with the 2000 estimated parameter values; \(\text{cvr}\) is the coverage rate defined as the percentage of the 2000 parameter estimates for which the true parameter value is contained in the \(\alpha\) confidence interval implied by the asymptotic distribution; \(S_0 = 100, \delta_0 = 0.2, T_3 = 504, r_0 = 0.05\) and \(N = 502\)

We note that the mean values of the estimates of \(\hat{\kappa}\) and \(\hat{\theta}\) are closer to the true values when we increase the sample size. Although we observe a notable reduction in the standard deviations of these estimates, they remain rather high compared to the other variables.

\(^2\) A coverage rate is a value reporting the percentage of the parameter estimates for which the true parameter value is contained in the \(\alpha\) confidence interval implied by the asymptotic distribution.
5 Real data application

In this section we apply the procedure outlined in section 4.2 using real data. The data used to estimate the parameters of domestic and foreign zero-coupon bonds are respectively futures contracts on three-month Eurodollar time deposits traded on the Chicago Mercantile Exchange (CME) and futures contracts on three-month Canadian bankers’ acceptances (BAX) traded in the Montreal Exchange. Each contract represents respectively a $1 million Eurodollar and a $1 million Canadian bankers’ acceptances maturing 90-days after contract expiration. Both BAX and Eurodollar futures contracts are settled cash and have the same delivery date which is the second London bank business day before the third Wednesday of the contract month, which rests in the March, June, September and December cycle. The Eurodollar futures contracts and the BAX are chosen for their extreme liquidity. Our sample consists of daily data ranging from January 1, 2000 to December 18, 2000 on prices of both Eurodollar futures contracts and BAX maturing in December 2000.

We apply the quadratic hill-climbing algorithm of Goldfeld, Quandt and Trotter (1996) to estimate the parameters $\hat{\sigma}_f$, $\hat{\sigma}_g$, and $\hat{\rho}_{34}$ that maximize the loglikelihood function (10). Supposing that these parameters are known, we have to estimate $\hat{\alpha}_S$, $\hat{\alpha}_C$, $\hat{\sigma}_S$, $\hat{\sigma}_C$, $\hat{\rho}_{12}$, $\hat{\rho}_{13}$, $\hat{\rho}_{14}$, $\hat{\rho}_{23}$, and $\hat{\rho}_{24}$ that maximize the global likelihood function (8). To get these estimated values, we used daily data on gold prices and USD/CAD exchange rates covering the period from January 1, 2000 to December 29, 2000.

Finally, to estimate the convenience yield as well as the parameters describing its evolution, we used daily data on gold prices and on gold-future prices expiring on December 2000. Our sample ranges from January 1, 2000 to December 29, 2000.

All the data used in this study is obtained from Datastream.

To be completed later.

6 Conclusion

In this paper we developed a theoretical model for pricing a basket option on commodity prices, exchange rates and zero-coupon bonds. Our contributions consist essentially in looking at a basket option based on multiple underlying assets which are intrinsically different and in considering all the aspects of basket options such as modelisation; basket-option pricing; and estimation of parameters. The empirical implementation of our model raises several problems. Many of the variables prove to be unobservable: variables such as the correlation between the underlying assets; the commodity’s convenience yield; the market price of convenience yield risk; and the market prices of risks related to zero-coupon bonds. To overcome these problems, we first supposed that the process describing the convenience yield shares the same source of risk as the commodity process. Second, we viewed the futures contract as a derivative instrument on instantaneous forward rate and thus as deriving its uncertainty from the same source as the forward rate. Though, we did not have to estimate the market price of risks on zero-coupon bonds.

Since the valuation model proposed depends on the finite sum of correlated lognormal random variables, we did not derive an analytic expression for the basket-option price. We instead used a
Monte Carlo simulation to value the basket option. The results show that the numerical solutions converge very well.

Finally, we applied the maximum likelihood method to estimate the parameters of risky assets. The Monte Carlo study reveals that the maximum likelihood estimates are unbiased and precise. As proposed by Duan (1994), we also adapted the maximum likelihood approach to unobservable variables, in order to estimate the convenience yield as well as the parameters describing its evolution. The results show that all the parameters estimates are convergent, except for the speed of the adjustment coefficient.

We intend to explore other issues related to basket options in subsequent research. For example, it would be very interesting to use analytical approximations to value basket option. Less accurate than Monte Carlo simulations, these approximations are faster and thus possibly more attractive to market practitioners. Another possible area of research would be the study of the hedging performance of the basket as compared to a set of individual options.
A The model under the risk neutral measure

This section derives the theoretical results presented in section 2. The equations describing the dynamics of the stochastic processes are expressed under the objective probability measure $\mathcal{P}$ and are given by equations (1a) - (1g). Because it is easier to work with independent Brownian motions, we use the following transformation:

$$
\begin{pmatrix}
W^{(1)} \\
W^{(2)} \\
W^{(3)} \\
W^{(4)}
\end{pmatrix}' = AB,
$$

where $A$ is the Cholesky decomposition of the correlation matrix of the four dependant Brownian motions and $B$ is a column vector containing the four independent Brownian motions. Since the covariance matrix is positive definite, we know that it is invertible and that $A$ exists. In order to change from the objective measure to the risk neutral measure, we need to build the financial assets available to the local investor in the domestic currency. In a first step, we define $Y_t^{(1)} = F_t C_t$, the value of the foreign bank account $F_t$ expressed in domestic currency. Second, we define $Y_t^{(2)}(t, T) = C_t K(t, T)$ which represents the value of the foreign zero-coupon bond $K(t, T)$ expressed in domestic currency. Finally, we define the process $Y_t^{(3)} = S_t \exp \left[ \int_0^t \delta_s ds \right]$ where $\delta_s$ denotes the commodity convenience yield. In fact, $Y_t^{(3)}$ is the value of a portfolio initially made up of a unit of the commodity $S_0$, and at every moment where the benefits are perceived, they will be used immediately to buy a little more commodity.

The present value of the financial assets available to the local investor in the domestic currency are thus:

$$
\begin{align*}
Z_t^{(1)} &= D_t^{-1} Y_t^{(1)} : \text{The present value of a foreign cash bond expressed in local currency;} \\
Z_t^{(2)} &= D_t^{-1} Y_t^{(2)}(t, T) : \text{The present value of a foreign zero-coupon bond expressed in local currency;} \\
Z_t^{(3)} &= D_t^{-1} Y_t^{(3)} : \text{The present value of the portfolio having as components the commodity and the benefits accruing from holding this commodity;} \\
Z_t^{(4)} &= D_t^{-1} P(t, T) : \text{The present value of a domestic zero coupon bond.}
\end{align*}
$$

Applying Itô lemma we obtain:
\[ dZ_t^{(1)} = Z_t^{(1)} \left[ (\alpha_C + u_t - r_t) dt + \sigma_C \sum_{i=1}^2 a_{2i} dB_t^{(i)} \right], \]  
\[ dZ_t^{(2)} = Z_t^{(2)} \left[ \left( \frac{1}{2} \sigma_g^2 (T-t)^2 - \int_t^T \alpha_g (t, u) du + g (0, t) + \alpha_C - \sigma_C^2 \sigma_g (T-t) - r_t \right) dt \right], \]  
\[ dZ_t^{(3)} = Z_t^{(3)} \left[ (\delta_t - r_t) dt + \sigma_S dB_t^{(1)} \right], \]  
\[ dZ_t^{(4)} = Z_t^{(4)} \left[ \frac{1}{2} \sigma_f^2 (T-t)^2 - \int_t^T \alpha_f (t, u) du + f (0, t) - r_t \right) dt - \sigma_f (T-t) \sum_{i=1}^3 a_{3i} dB_t^{(i)} \right]. \]  

Using the Cameron-Martin-Girsanov theorem, it is possible to show the existence of a unique risk-neutral measure, denoted \( Q \), such that the present values of the tradable securities  
\[ \{ Z_t^{(1)} = D_t^{-1} Y_t^{(1)} : t \geq 0 \}, \{ Z_t^{(2)} = D_t^{-1} Y_t^{(2)} (t, T) : t \geq 0 \}, \{ Z_t^{(3)} = D_t^{-1} Y_t^{(3)} : t \geq 0 \} \]  
and  \[ \{ Z_t^{(4)} = D_t^{-1} P(t, T) : t \geq 0 \} \]  
are \( Q \)-martingales. Indeed, fixing  
\[ \tilde{B}_t^{(i)} = B_t^{(i)} + \gamma_{it} \quad \text{for} \quad 0 \leq t \leq T \quad \text{and} \quad i \in \{1, 2, 3, 4\}, \]  
where  
\[ \gamma_{t}^{(1)} = \frac{\alpha_S + \delta_t - r_t}{\sigma_S}, \]  
\[ \gamma_{t}^{(2)} = \frac{\sigma_S (\alpha_C + u_t - r_t) - \sigma_C a_{21} (\alpha_S + \delta_t - r_t)}{\sigma_S a_{22}}, \]  
\[ \gamma_{t}^{(3)} = \left[ \frac{-1}{2} \frac{\sigma_f (T-t)}{a_{33}} + \int_t^T \frac{\alpha_f (t, u) du}{\sigma_f (T-t) a_{33}} + \frac{r_t}{\sigma_f (T-t) a_{33}} - \frac{a_{31}}{a_{33}} \gamma_{t}^{(1)} + \frac{a_{32}}{a_{33}} \gamma_{t}^{(2)} \right], \]  
\[ \gamma_{t}^{(4)} = \left[ \frac{-1}{2} \frac{\sigma_g (T-t)}{a_{44}} + \int_t^T \frac{\alpha_g (t, u) du}{\sigma_g (T-t) a_{44}} + \frac{\sigma_C a_{24}}{a_{44}} + \frac{\sigma_C (a_{21} \gamma_{t}^{(1)} + a_{22} \gamma_{t}^{(2)})}{\sigma_g (T-t) a_{44}} - \frac{a_{41} \gamma_{t}^{(1)}}{a_{44}} + \frac{a_{42} \gamma_{t}^{(2)}}{a_{44}} + \frac{a_{43} \gamma_{t}^{(3)}}{a_{44}} - \frac{a_{41} \gamma_{t}^{(1)}}{a_{44}} \right], \]  
then a measure exists, \( Q \), under which \( \tilde{B}_t^{(i)} \) are independent Brownian motions and
\[ dZ_t^{(1)} = Z_t^{(1)} \left[ \sigma_C \sum_{i=1}^{2} a_{2i} d\bar{B}_t^{(i)} \right], \]
\[ dZ_t^{(2)} = Z_t^{(2)} \left[ -\sigma_g (T - t) \sum_{i=1}^{4} a_{4i} d\bar{B}_t^{(i)} + \sigma_C \sum_{i=1}^{2} a_{2i} d\bar{B}_t^{(i)} \right], \]
\[ dZ_t^{(3)} = Z_t^{(3)} \left[ \sigma_s d\bar{B}_t^{(1)} \right], \]
\[ dZ_t^{(4)} = Z_t^{(4)} \left[ -\sigma_f (T - t) \sum_{i=1}^{3} a_{3i} d\bar{B}_t^{(i)} \right]. \]

Finally, under the risk-neutral probability measure \( Q \), the SDE satisfied by the different assets available in our complete market are given by,

\[ dS_t = (r_t - \delta_t)S_t \, dt + \sigma_S S_t \, d\bar{W}_t^{(1)}, \]
\[ d\delta_t = a(m - \delta_t)dt + \sigma_s d\bar{W}_t^{(1)}, \]
\[ dC_t = \left[ (r_t - u_t)C_t \, dt + \sigma_C C_t \, d\bar{W}_t^{(2)} \right], \]
\[ dP(t,T) = P(t,T) \left[ r_t \, dt - \sigma_f (T - t) \, d\bar{W}_t^{(3)} \right], \]
\[ dK(t,T) = K(t,T) \left[ [u_t + \sigma_g \sigma_C (T - t)] \, dt - \sigma_g (T - t) \, d\bar{W}_t^{(4)} \right], \]

where
\[ \left( \bar{W}^{(1)} \bar{W}^{(2)} \bar{W}^{(3)} \bar{W}^{(4)} \right)' = \left( W^{(1)} W^{(2)} W^{(3)} W^{(4)} \right)' + A\bar{\gamma} = A\bar{\gamma}. \]

**B  Change to the T-forward measure**

We have already defined in Appendix A the transformation:

\[ \left( \bar{W}^{(1)} \bar{W}^{(2)} \bar{W}^{(3)} \bar{W}^{(4)} \right)' = A\bar{B}_t, \]

where \( A \) is the Cholesky decomposition of the correlation matrix of the four dependant Brownian motions and \( \bar{B}_t \) is a column vector containing the four independent Brownian motions under the martingale measure \( Q \). If we consider that \( A_i \) represents the row \( i \) of matrix \( A \), then the SDE satisfied by the different assets under the risk neutral measure \( Q \) can be written as:
\[ dS_t = (r_t - \delta_t)S_t \, dt + \sigma_S S_t \, A_1 \, d \tilde{B}_t, \]
\[ d\delta_t = a(m - \delta_t)dt + \sigma_\delta \, A_1 \, d \tilde{B}_t, \]
\[ dC_t = \left[ (r_t - u_t)C_t \, dt + \sigma_C C_t \, A_2 \right] \, d \tilde{B}_t, \]
\[ dP(t, T) = P(t, T) \left[ r_t \, dt - \sigma_f (T - t) \, A_3 \right] \, d \tilde{B}_t, \]
\[ dK(t, T) = K(t, T) \left[ [u_t + \sigma_g \sigma_C (T - t)] \, dt - \sigma_g (T - t) \, A_4 \right] \, d \tilde{B}_t, \]

The \( Q \) measure represents the equivalent risk-neutral measure corresponding to the domestic bank account numeraire \( e^{\int_t^T r_u \, du} \) while \( Q_T \) is defined as the T-forward measure corresponding to the T-bond price numeraire \( P(t, T) \). Thus the Girsanov transformation from the risk-neutral measure to the T-forward measure is defined below

\[ d \tilde{B}_t = d \tilde{B}_t + \sigma_f (T - t) \, A_3 \, dt, \]

where \( \tilde{B}_t \) is the four-dimensional independent Brownian motions under the T-forward measure \( Q_T \). As the correlation structure of the Brownian motions, whether under the risk neutral measure \( Q \) or the T-forward measure \( Q_T \) remains the same and using the same computational techniques as in the Appendix A, the SDE satisfied by the different risky assets under the T-forward probability measure can be defined as below:

\[ dS_t = \left[ r_t - \delta_t - \sigma_S \sigma_f \rho_{13} (T - t) \right] S_t \, dt + \sigma_S S_t \, d \tilde{W}^{(1)}_t, \]  \hspace{1cm} (18a)
\[ d\delta_t = a \left[ m - \delta_t - \sigma_\delta \sigma_f \rho_{13} (T - t) \right] dt + \sigma_\delta \, d \tilde{W}^{(1)}_t, \]  \hspace{1cm} (18b)
\[ dC_t = \left[ r_t - u_t - \sigma_C \sigma_f \rho_{23} (T - t) \right] C_t \, dt + \sigma_C C_t \, d \tilde{W}^{(2)}_t, \]  \hspace{1cm} (18c)
\[ dP(t, T) = \left[ r_t + \sigma_f^2 (T - t)^2 \right] P(t, T) \, dt - \sigma_f (T - t) \, P(t, T) \, d \tilde{W}^{(3)}_t, \]  \hspace{1cm} (18d)
\[ dK(t, T) = \left[ [u_t + \sigma_g \sigma_C (T - t) + \sigma_g \sigma_f \rho_{34} (T - t)] \, K(t, T) \right] dt \]  \hspace{3.5cm} - \sigma_g (T - t) \, K(t, T) \, d \tilde{W}^{(4)}_t \]  \hspace{1cm} (18e)

The strong solution of the system of equations (18) under the T-forward measure \( Q_T \) is given respectively by,
where

de
determine the values of these terms. By de

Recall that the vector of risky assets

\[ \mu_Q \]

is given by

\[ \mathbf{1}^T \exp \left\{ \mathbf{1}^T \mathbf{W}_u^{(1)} \right\} \],

where \( m_1 = m - \frac{1}{2} \sigma \delta \sigma f \rho_{13} (T - t) \).

C The multivariate distribution of risky assets under the T-forward measure

Recall that the vector of risky assets \( (S_T, P(T, T_1), Y(T, T_1)) \) under the new martingale measure follow a multivariate distribution and its density function is given by

\[ f(s, p, y) = \frac{1}{sp\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} \left( x - \mu_{QT} \right)^T \Sigma_{QT} \left( x - \mu_{QT} \right) \right\}, \]

where \( x \) is the vector of realized returns of risky assets

\[ x = \begin{pmatrix} \ln \left( \frac{s_T}{s_t} \right) \\ \ln \left( \frac{P(T, T_1)}{p(T, T_1)} \right) \\ \ln \left( \frac{Y(T, T_1)}{y(T, T_1)} \right) \end{pmatrix}, \]

\( \mu_{QT} \) represents the expected returns of risky assets under the T-forward measure \( Q_T \) and \( \Sigma_{QT} \) is the variance-covariance matrix under the new martingale measure \( Q_T \). In the following we will determine the values of these terms. By definition, we have

\[ \mu_{QT} = \left( \begin{array}{c} E_{QT} \left[ \ln \left( \frac{s_T}{s_t} \right) | \mathcal{F}_t \right] \\ E_{QT} \left[ \ln \left( \frac{P(T, T_1)}{p(T, T_1)} \right) | \mathcal{F}_t \right] \\ E_{QT} \left[ \ln \left( \frac{Y(T, T_1)}{y(T, T_1)} \right) | \mathcal{F}_t \right] \end{array} \right). \]
Using the equation (19a) in Appendix B of the commodity price $S_T$ we can calculate

$$E_{Q_T} \left[ \ln \left( \frac{S_T}{S_t} \right) | F_t \right] = \left[ \int_t^T f(0,u) \, du - (m_1 + \frac{1}{2} \sigma^2_S)(T-t) + (m_1 - \delta_t) \left( \frac{1-e^{-a(T-t)}}{a} \right) \right].$$

Using the strong solution of the domestic zero-coupon bond given by equation (19d) in Appendix B, we can determine

$$E_{Q_T} \left[ \ln(P_{T,T_1}) | F_t \right] = \left[ - \int_T^{T_1} f(0,u) \, du - \frac{1}{6} \sigma^2_f (T^3 - T^3) \right].$$

Finally, using the strong solutions of the exchange rate and foreign zero-coupon bond given by equations (19c) and (19e) respectively in Appendix B, we can determine the value of a foreign bond converted in domestic currency, in fact $Y(T,T_1) = K(T,T_1) \times C_T$ is given by:

$$Y(T,T_1) = C_t \exp \left[ \int_t^T f(0,u) \, du - \int_t^{T_1} g(0,u) \, du - \frac{1}{2} \sigma^2_C(T-t) + \frac{1}{2} \sigma_g \sigma_C (T^2 - t^2) \right] + \frac{1}{6} \left( \sigma^2_f - \sigma^2_g \right) (T^3 - t^3) + \left( \sigma_g \sigma_C T - \frac{1}{2} \sigma^2_g T T_1 \right) (T_1 - T) - \frac{1}{2} \sigma_C \sigma_f \rho_{23} (T-t)^2 - \frac{1}{2} \sigma_g \sigma_f \rho_{34} (T_1 - T)^2 + \sigma_C \int_t^T d \tilde{W}_u^{(2)} + \sigma_f \int_t^T d \tilde{W}_u^{(3)} - \sigma_g \int_t^T d \tilde{W}_u^{(4)} - \sigma_g \tilde{W}_T^{(4)} (T_1 - T) \right].$$

thus we can compute

$$E_{Q_T} \left[ \ln \left( Y(T,T_1) \right) | F_t \right] = \left[ \int_t^T f(0,u) \, du - \int_t^{T_1} g(0,u) \, du - \frac{1}{2} \sigma^2_C(T-t) + \frac{1}{2} \sigma_g \sigma_C (T^2 - t^2) \right] + \frac{1}{6} \left( \sigma^2_f - \sigma^2_g \right) (T^3 - t^3) + \left( \sigma_g \sigma_C T - \frac{1}{2} \sigma^2_g T T_1 \right) (T_1 - T) - \frac{1}{2} \sigma_C \sigma_f \rho_{23} (T-t)^2 - \frac{1}{2} \sigma_g \sigma_f \rho_{34} (T_1 - T)^2 + \ln (C_t) \right].$$

Now, we have to determine the different elements of the variance-covariance matrix. The variance of the commodity return is given by

$$\text{Var} 1 = \text{Var}_{Q_T} \left[ \ln \left( \frac{S_T}{S_t} \right) | F_t \right] \ \ \ \ \ \ \ = E_{Q_T} \left\{ \left[ \ln \left( \frac{S_T}{S_t} \right) - E_{Q_T} \left[ \ln \left( \frac{S_T}{S_t} \right) | F_t \right] \right]^2 | F_t \right\} \ \ \ = E_{Q_T} \left\{ \left[ \left( \sigma_f \int_t^T d \tilde{W}_u^{(3)} + \int_t^T \left( \sigma S - \sigma_S \left( \frac{1-e^{-a(T-u)}}{a} \right) \right) \, d \tilde{W}_u^{(1)} \right) \right]^2 \right\} \ \ \ \ \ = \ \ \ \ \ \ \ = \left[ \int_t^T \sigma_f^2 du + \int_t^T \left( \sigma_S - \sigma_S \left( \frac{1-e^{-a(T-u)}}{a} \right) \right)^2 \, du \right] + \frac{1}{2} \sigma_f \sigma_f \rho_{23} (T-t)^2 \left[ \int \tilde{W}_u^{(1)} d \tilde{W}_u^{(3)} \right] \ \ \ \ \ + \ \ \ \ \ \ \ \ \ \ \ + \left[ \sigma^2_f + \left( \sigma^2_S - \frac{2 \sigma_S \rho_{34}}{a} \right) + \frac{\sigma^2_f}{a^2} \right] + 2 \rho_{13} \left( \sigma_f \sigma_S - \frac{\sigma_f \rho_{34}}{a} \right) \left( \sigma_f \sigma_S - \frac{\sigma_f \rho_{34}}{a} \right) \right].$$
Finally we have to determine the covariances between the different risky assets composing the basket. Thus,

\[
\text{Cov12} = \text{Cov}_{Q_f} \left[ \ln \left( \frac{S_T}{S_t} \right), \ln (P(T,T_1)) | \mathcal{F}_t \right]
= \text{Cov}_{Q_f} \left[ \int_t^T d \tilde{W}^{(2)}_u + \int_t^T d \tilde{W}^{(3)}_u, \sigma_S - \sigma \delta \left( \frac{1 - e^{-a(T-u)}}{a} \right) \right] \tilde{W}^{(1)}_u, -\sigma \tilde{W}^{(3)}_T (T_1 - T) | \mathcal{F}_t
\]

= \sigma_f (T_1 - T) \left[ - \left( \sigma_f + \rho_{13} \left( \sigma_S - \frac{\sigma \delta}{a} \right) \right) (T - t) - \rho_{13} \frac{\sigma \delta}{a} \left( 1 - e^{-a(T-t)} \right) \right],

\[
\text{Cov13} = \text{Cov}_{Q_f} \left[ \ln \left( \frac{S_T}{S_t} \right), \ln (Y(T,T_1)) | \mathcal{F}_t \right]
= \text{Cov}_{Q_f} \left[ \sigma_S - \sigma \delta \left( \frac{1 - e^{-a(T-u)}}{a} \right) \right] \tilde{W}^{(1)}_u, -\sigma \tilde{W}^{(3)}_T (T_1 - T) | \mathcal{F}_t
\]

= \left[ \sigma_f^2 + \sigma_g^2 + \sigma_C^2 + 2 \sigma_f \sigma_C \rho_{23} + 2 \sigma_g^2 (T_1 - T) - (2 \sigma_g \sigma_C \rho_{24} + 2 \sigma_f \sigma_g \rho_{34}) (1 + (T_1 - T)) \right] (T - t) + \sigma_g^2 (T_1 - T)^2.

Thus, the variance-covariance matrix is given by
is a Martingale under the risk-neutral measure domestic and foreign zero-coupon bonds respectively. The futures contracts mature at time \( T \).

Denote by \( D \) the evolution of futures price on zero-coupon bonds respectively, we obtain,

\[
F_D(t, T_1, T_2) = \mathbb{E}_Q [F_D(T_1, T_1, T_2) \mid \mathcal{F}_t] = \mathbb{E}_Q [P(T_1, T_2) \mid \mathcal{F}_t],
\]

(22)

\[
F_F(t, T_1, T_2) = \mathbb{E}_Q [F_F(T_1, T_1, T_2) \mid \mathcal{F}_t] = \mathbb{E}_Q [K(T_1, T_2) \mid \mathcal{F}_t].
\]

(23)

Recall that under the equivalent measure \( Q \), the domestic and foreign zero-coupon bonds at time \( T_1 \) and maturing at time \( T_2 \) are given by

\[
P(T_1, T_2) = \exp \left[ -\int_{t}^{T_2} f(0, u)du - \frac{1}{2}\sigma_f^2 T_1 T_2 (T_2 - T_1) - \sigma_f(T_2 - T_1) W_{T_1}^{(3)} \right],
\]

\[
K(T_1, T_2) = \exp \left[ -\int_{T_1}^{T_2} g(0, u)du - \frac{1}{2}\sigma_g^2 T_1 T_2 (T_2 - T_1) + \sigma_g \sigma_C T_1 (T_2 - T_1) - \sigma_g(T_2 - T_1) W_{T_1}^{(4)} \right],
\]

If we substitute the expressions of \( P(T_1, T_2) \) and \( K(T_1, T_2) \) in the equations (22) and (23) respectively, we obtain,

\[
F_D(t, T_1, T_2) = \exp \left[ -\int_{t}^{T_2} f(0, u)du - \frac{1}{2}\sigma_f^2 T_1 T_2 (T_2 - T_1) - \sigma_f(T_2 - T_1) W_{t}^{(3)} + \frac{1}{2}\sigma_f^2 (T_2 - T_1)^2 (T_1 - t) \right],
\]

\[
F_F(t, T_1, T_2) = \exp \left[ -\int_{T_1}^{T_2} g(0, u)du - \frac{1}{2}\sigma_g^2 T_1 T_2 (T_2 - T_1) + \sigma_g \sigma_C T_1 (T_2 - T_1) - \sigma_g(T_2 - T_1) W_t^{(4)} + \frac{1}{2}\sigma_g^2 (T_2 - T_1)^2 (T_1 - t) \right].
\]

If we suppose that futures prices are observed at different periods of time \( t_1, t_2...t_{i-1}, t_i...t_N = T_1 \) such that \( t_i - t_{i-1} = \Delta_t \), we get

\[
F_D(t_i, T_1, T_2) = F_D(t_{i-1}, T_1, T_2) \exp \left[ -\frac{1}{2}\sigma_f^2 (T_2 - T_1)^2 \Delta_t - \sigma_f(T_2 - T_1) \left( W_{t_i}^{(3)} - W_{t_{i-1}}^{(3)} \right) \right],
\]

\[
F_F(t_i, T_1, T_2) = F_F(t_{i-1}, T_1, T_2) \exp \left[ -\frac{1}{2}\sigma_g^2 (T_2 - T_1)^2 \Delta_t - \sigma_g(T_2 - T_1) \left( W_{t_i}^{(4)} - W_{t_{i-1}}^{(4)} \right) \right].
\]
E Global likelihood function

In the following, we will suppose that assets are observed at different periods of time $t_1, t_2...t_{i−1}$, $t_i...t_N$ such that $t_i - t_{i−1} = \Delta t$.

In order to determine the global likelihood function, we have to determine first the parameters of the joint distribution of risky assets under the probability measure $P$. The expected returns of commodity, exchange rate, futures contracts on domestic and foreign zero-coupon bonds are respectively,

\[
E_P \left[ \ln \left( \frac{S_t}{S_{t−1}} \right) \mid \mathcal{F}_{t−1} \right] = E_P \left[ \left( \alpha_S - \frac{1}{2} \sigma^2_S \right) \Delta t + \sigma_S \left( W_{t_i}^{(1)} - W_{t_{i−1}}^{(1)} \right) \mid \mathcal{F}_{t−1} \right] = \left( \alpha_S - \frac{1}{2} \sigma^2_S \right) \Delta t,
\]

\[
E_P \left[ \ln \left( \frac{C_t}{C_{t−1}} \right) \mid \mathcal{F}_{t−1} \right] = E_P \left[ \left( \alpha_C - \frac{1}{2} \sigma^2_C \right) \Delta t + \sigma_C \left( W_{t_i}^{(2)} - W_{t_{i−1}}^{(2)} \right) \mid \mathcal{F}_{t−1} \right] = \left( \alpha_C - \frac{1}{2} \sigma^2_C \right) \Delta t,
\]

\[
E_P \left[ \ln \left( \frac{F_D (t_i, T_1, T_2)}{F_D (t_{i−1}, T_1, T_2)} \right) \mid \mathcal{F}_{t−1} \right] = E_P \left[ \left( \alpha_F - \frac{1}{2} \sigma^2_F \right) \Delta t + \sigma_F \left( W_{t_i}^{(3)} - W_{t_{i−1}}^{(3)} \right) \mid \mathcal{F}_{t−1} \right] = \frac{1}{2} \sigma^2_F (T_2 - T_1)^2 \Delta t,
\]

and

\[
E_P \left[ \ln \left( \frac{F_F (t_i, T_1, T_2)}{F_F (t_{i−1}, T_1, T_2)} \right) \mid \mathcal{F}_{t−1} \right] = E_P \left[ \left( \alpha_F - \frac{1}{2} \sigma^2_F \right) \Delta t + \sigma_F \left( W_{t_i}^{(4)} - W_{t_{i−1}}^{(4)} \right) \mid \mathcal{F}_{t−1} \right] = \frac{1}{2} \sigma^2_F (T_2 - T_1)^2 \Delta t.
\]
Moreover, the variances of commodity return, exchange rate, futures on domestic and foreign zero coupon bonds are respectively,

$$\text{Var}_P \left[ \ln \left( \frac{S_{t_i}}{S_{t_{i-1}}} \right) \mid \mathcal{F}_{t_{i-1}} \right] = \sigma_S^2 \Delta t,$$

$$\text{Var}_P \left[ \ln \left( \frac{C_{t_i}}{C_{t_{i-1}}} \right) \mid \mathcal{F}_{t_{i-1}} \right] = \sigma_C^2 \Delta t,$$

$$\text{Var}_P \left[ \ln \left( \frac{F_D(t_i, T_1, T_2)}{F_D(t_{i-1}, T_1, T_2)} \right) \mid \mathcal{F}_{t_{i-1}} \right] = \sigma_f^2 (T_2 - T_1)^2 \Delta t,$$

$$\text{Var}_P \left[ \ln \left( \frac{F_F(t_i, T_1, T_2)}{F_F(t_{i-1}, T_1, T_2)} \right) \mid \mathcal{F}_{t_{i-1}} \right] = \sigma_g^2 (T_2 - T_1)^2 \Delta t.$$

Finally, the covariances between the different assets in the basket are given by,

$$\text{Cov}_P \left[ \ln \left( \frac{S_{t_i}}{S_{t_{i-1}}} \right), \ln \left( \frac{C_{t_i}}{C_{t_{i-1}}} \right) \mid \mathcal{F}_{t_{i-1}} \right]$$

$$= \text{Cov}_P \left[ \left( \alpha_S - \frac{1}{2} \sigma_S^2 \right) \Delta t + \sigma_S \left( W_{t_i}^{(1)} - W_{t_{i-1}}^{(1)} \right), \left( \alpha_C - \frac{1}{2} \sigma_C^2 \right) \Delta t + \sigma_C \left( W_{t_i}^{(2)} - W_{t_{i-1}}^{(2)} \right) \mid \mathcal{F}_{t_{i-1}} \right]$$

$$= \sigma_S \sigma_C \Delta t \rho_{12},$$

$$\text{Cov}_P \left[ \ln \left( \frac{S_{t_i}}{S_{t_{i-1}}} \right), \ln \left( \frac{F_D(t_i, T_1, T_2)}{F_D(t_{i-1}, T_1, T_2)} \right) \mid \mathcal{F}_{t_{i-1}} \right]$$

$$= \text{Cov}_P \left[ \left( \alpha_S - \frac{1}{2} \sigma_S^2 \right) \Delta t + \sigma_S \left( W_{t_i}^{(1)} - W_{t_{i-1}}^{(1)} \right), -\frac{1}{2} \sigma_f^2 (T_2 - T_1)^2 \Delta t \right]$$

$$= -\sigma_S \sigma_f (T_2 - T_1) \Delta t \rho_{13},$$

$$\text{Cov}_P \left[ \ln \left( \frac{S_{t_i}}{S_{t_{i-1}}} \right), \ln \left( \frac{F_F(t_i, T_1, T_2)}{F_F(t_{i-1}, T_1, T_2)} \right) \mid \mathcal{F}_{t_{i-1}} \right]$$

$$= \text{Cov}_P \left[ \left( \alpha_S - \frac{1}{2} \sigma_S^2 \right) \Delta t + \sigma_S \left( W_{t_i}^{(1)} - W_{t_{i-1}}^{(1)} \right), -\frac{1}{2} \sigma_g^2 (T_2 - T_1)^2 \Delta t \right]$$

$$= -\sigma_S \sigma_g (T_2 - T_1) \Delta t \rho_{14},$$
Hence, the joint distribution of the vector
\[\begin{align*}
\text{Cov}_P \left[ \ln \left( \frac{C_i}{C_{i-1}} \right), \ln \left( \frac{D_i(t_i, T_1, T_2)}{D_i(t_{i-1}, T_1, T_2)} \right) \mid \mathcal{F}_{t_{i-1}} \right] \\
= \text{Cov}_P \left[ \left( \alpha C - \frac{1}{2} \sigma_C^2 \right) \Delta_t + \sigma_C \left( \tilde{W}_{t_i}^{(2)} - \tilde{W}_{t_{i-1}}^{(2)} \right), -\frac{1}{2} \sigma_f^2 (T_2 - T_1)^2 \Delta_t \right] \\
= -\sigma_C \sigma_f (T_2 - T_1) \Delta_t \rho_{23},
\end{align*}\]

\[\begin{align*}
\text{Cov}_P \left[ \ln \left( \frac{C_i}{C_{i-1}} \right), \ln \left( \frac{F_i(t_i, T_1, T_2)}{F_i(t_{i-1}, T_1, T_2)} \right) \mid \mathcal{F}_{t_{i-1}} \right] \\
= \text{Cov}_P \left[ \left( \alpha C - \frac{1}{2} \sigma_C^2 \right) \Delta_t + \sigma_C \left( \tilde{W}_{t_i}^{(2)} - \tilde{W}_{t_{i-1}}^{(2)} \right), -\frac{1}{2} \sigma_g^2 (T_2 - T_1)^2 \Delta_t \right] \\
= -\sigma_C \sigma_g (T_2 - T_1) \Delta_t \rho_{24},
\end{align*}\]

\[\begin{align*}
\text{Cov}_P \left[ \ln \left( \frac{D_i(t_i, T_1, T_2)}{D_i(t_{i-1}, T_1, T_2)} \right), \ln \left( \frac{F_i(t_i, T_1, T_2)}{F_i(t_{i-1}, T_1, T_2)} \right) \mid \mathcal{F}_{t_{i-1}} \right] \\
= \text{Cov}_P \left[ -\frac{1}{2} \sigma_f^2 (T_2 - T_1)^2 \Delta_t - \sigma_f (T_2 - T_1) \left( \tilde{W}_{t_i}^{(3)} - \tilde{W}_{t_{i-1}}^{(3)} \right), \\
-\frac{1}{2} \sigma_g^2 (T_2 - T_1)^2 \Delta_t - \sigma_g (T_2 - T_1) \left( \tilde{W}_{t_i}^{(4)} - \tilde{W}_{t_{i-1}}^{(4)} \right) \mid \mathcal{F}_{t_{i-1}} \right] \\
= \sigma_f \sigma_g (T_2 - T_1)^2 \rho_{34} \Delta_t.
\end{align*}\]

Hence, the joint distribution of the vector
\[X_{t_i} = \left( \begin{array}{c}
\ln \left( \frac{S_i}{S_{i-1}} \right) \\
\ln \left( \frac{C_i}{C_{i-1}} \right) \\
\ln \left( \frac{D_i(t_i, T_1, T_2)}{D_i(t_{i-1}, T_1, T_2)} \right) \\
\ln \left( \frac{F_i(t_i, T_1, T_2)}{F_i(t_{i-1}, T_1, T_2)} \right)
\end{array} \right),\]
given the information available at time \(t_{i-1}\) is
\[
(X_{t_i} \mid x_{t_{i-1}}, x_{t_{i-2}}, \ldots, x_2) \sim N \left( \mu_P, \Sigma_P \right),
\]

34
where \( x_t = (\ln \left( \frac{s_t}{s_{t-1}} \right), \ln \left( \frac{c_t}{c_{t-1}} \right), \ln \left( f_{D(t_i,T_1,T_2)} \right), \ln \left( f_{F(t_i,T_1,T_2)} \right), \ln \left( f_{F(t_i,T_1,T_2)} \right) \)' represents the realized returns of risky assets, \((s_t, c_t, f_D(t_i, T_1, T_2), f_F(t_i, T_1, T_2))'\) is the realization of the random vector \((S_t, C_t, F_D(t_i, T_1, T_2), F_F(t_i, T_1, T_2))'\),

\[
\mu_P = \begin{pmatrix}
(\alpha_S - \frac{1}{2}\sigma^2_S)\Delta_t \\
(\alpha_C - \frac{1}{2}\sigma^2_C)\Delta_t \\
-\frac{1}{2}\sigma_f^2(T_2 - T_1)^2\Delta_t \\
-\frac{1}{2}\sigma_g^2(T_2 - T_1)^2\Delta_t
\end{pmatrix},
\]

(24)

and

\[
\Sigma_P = \begin{pmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\
\sigma_{12} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\
\sigma_{13} & \sigma_{23} & \sigma_{33} & \sigma_{34} \\
\sigma_{14} & \sigma_{24} & \sigma_{34} & \sigma_{44}
\end{pmatrix},
\]

(25)

where

\[
\begin{align*}
\sigma_{11} & = \sigma^2_S\Delta_t, \\
\sigma_{22} & = \sigma^2_C\Delta_t, \\
\sigma_{33} & = \sigma_f^2(T_2 - T_1)^2\Delta_t, \\
\sigma_{44} & = \sigma_g^2(T_2 - T_1)^2\Delta_t, \\
\sigma_{12} & = \sigma_S\sigma_C\Delta_t\rho_{12}, \\
\sigma_{13} & = -\sigma_S\sigma_f(T_2 - T_1)\Delta_t\rho_{13}, \\
\sigma_{14} & = -\sigma_S\sigma_g(T_2 - T_1)\Delta_t\rho_{14}, \\
\sigma_{23} & = -\sigma_C\sigma_f(T_2 - T_1)\Delta_t\rho_{23}, \\
\sigma_{24} & = -\sigma_C\sigma_g(T_2 - T_1)\Delta_t\rho_{24}, \\
\sigma_{34} & = \sigma_f\sigma_g(T_2 - T_1)^2\Delta_t\rho_{34}.
\end{align*}
\]

Because the returns on risky assets are markovian processes, the vector of risky assets follows a lognormal multivariate conditional distribution and its conditional density function is given by,

\[
f(s_{t_i}, c_{t_i}, f_D(t_i, T_1, T_2), f_F(t_i, T_1, T_2) | s_{t_i-1}, c_{t_i-1}, f_D(t_i-1, T_1, T_2), f_F(t_i-1, T_1, T_2)) = \frac{1}{\sqrt{2\pi}} \times \frac{1}{\sqrt{||\Sigma_P||}} \times \exp \left[ -\frac{1}{2} (x_t - \mu_P)'\Sigma_P^{-1}(x_t - \mu_P) \right],
\]

where \( \mu_P \) represents the vector of expected returns of risky assets given by the equation (24) and \( \Sigma_P \) represents the variance-covariance matrix of risky assets’ returns which is given by the equation
Since the multivariate likelihood function of risky assets is the product of joint density functions, we get
\[
L(\mathbf{J}, \mu_p, \Sigma_P) = \prod_{i=2}^{N} \frac{1}{s_{t_i} c_{t_i} f_{t_i}^{P} F_{t_i}} \times \left( \frac{1}{\sqrt{2\pi}} \right)^{N-1} \times \left( \frac{1}{\sqrt{\|\Sigma_P\|}} \right)^{N-1} \times \prod_{i=2}^{N} \exp \left[ -\frac{1}{2} (x_{t_i} - \mu_P)' \Sigma_P^{-1} (x_{t_i} - \mu_P) \right],
\]
where \( J = \{ J_{t_i} = (S_{t_i} C_{t_i} D_{t_i} (t_i, T_1, T_2) F_{t_i} (t_i, T_1, T_2)) : i = (1, 2, ..., N) \} \).

**F Value of the commodity future contract**

Hull (1997) demonstrated that the future contract price is equal to the expected price, under the martingale measure \( Q \), of the underlying asset at the maturity of the future contract. Though,
\[
F(t, T_3) = E_Q [S_{T_3} \mid \mathcal{F}_t] = E_Q [\exp (\ln (S_{T_3})) \mid \mathcal{F}_t].
\]

Knowing that the commodity return follows a normal distribution, we can write
\[
F(t, T_3) = \exp \left[ E_Q (\ln (S_{T_3})) \mid \mathcal{F}_t \right] + \frac{1}{2} \text{Var}_Q (\ln (S_{T_3})) \mid \mathcal{F}_t].
\]

Using the expressions of the mean and the variance of the commodity return under the risk neutral measure \( Q \) calculated in Appendix C, we can determine the price of the commodity future contract as
\[
F(t, T_3) = S_t \exp \left\{ \int_t^{T_3} f(0, u)du + \frac{1}{6} \sigma_f^2 (T_3^2 - t^2) + \frac{\sigma_f^2}{4a} \left( \frac{1-e^{-2a(T_3-t)}}{a} \right) \right\} + \left[ m + \frac{\sigma_s^2}{2a} + \frac{\sigma_s\sigma_a}{a} + \frac{1}{2} \sigma_f^2 + \rho_{13} \sigma_f (\sigma_s - \sigma_a) \right] (T_3 - t) \left[ \frac{1-e^{-a(T_3-t)}}{a} \right] + \left[ \frac{\sigma_s}{a} (\sigma_s - \frac{\sigma_s\sigma_a}{a} + \rho_{13} \sigma_f) + (m - \delta_t) \right] \left( \frac{1-e^{-a(T_3-t)}}{a} \right).
\]

**G Likelihood function of the convenience yield**

We can calculate the conditional mean and variance of \( \delta_{t_i} \) given the information available at time \( t_{i-1} \) like this,
\[ E_P [\delta_{t_i} | \mathcal{F}_{t_{i-1}}] \]
\[ = E_P [\delta_{t_{i-1}} e^{-\kappa \Delta t} + \theta (1 - e^{-\kappa \Delta t}) + \sigma_\delta \int_0^{\Delta t} e^{\kappa (u - \Delta t)} dW_u^{(1)}] \]
\[ = \delta_{t_{i-1}} e^{-\kappa \Delta t} + \theta (1 - e^{-\kappa \Delta t}), \]

and

\[ \text{Var}_P [\delta_{t_i} | \mathcal{F}_{t_{i-1}}] \]
\[ = \text{Var}_P [\delta_{t_{i-1}} e^{-\kappa \Delta t} + \theta (1 - e^{-\kappa \Delta t}) + \sigma_\delta \int_0^{\Delta t} e^{\kappa (u - \Delta t)} dW_u^{(1)}] \]
\[ = \frac{\sigma_\delta^2}{2\kappa} [1 - e^{-2\kappa \Delta t}] . \]

It appears clearly that \( \delta_{t_i} \) follows a normal conditional distribution and its conditional density function is given by

\[ f_{\delta_{t_i} | \delta_{t_{i-1}}} = \frac{\sqrt{2\kappa}}{\sqrt{2\pi \sigma_\delta^2 (1 - e^{-2\kappa \Delta t})}} \exp \left[ -\frac{\kappa (\delta_{t_{i}} - \delta_{t_{i-1}} e^{-\kappa \Delta t} - \theta (1 - e^{-\kappa \Delta t}))^2}{\sigma_\delta^2 (1 - e^{-2\kappa \Delta t})} \right] . \]

Therefore, we can express the multivariate likelihood function of the convenience yield as the product of the conditional density function,

\[ L (\delta; \kappa, \theta, \sigma_\delta) = \left( \frac{1}{\sqrt{2\pi}} \right)^{N-1} \times \left( \frac{\sqrt{2\kappa}}{\sigma_\delta \sqrt{1 - e^{-2\kappa \Delta t}}} \right)^{N-1} \times \prod_{i=2}^{N} \exp \left[ -\frac{\kappa (\delta_{t_{i}} - \delta_{t_{i-1}} e^{-\kappa \Delta t} - \theta (1 - e^{-\kappa \Delta t}))^2}{\sigma_\delta^2 (1 - e^{-2\kappa \Delta t})} \right] . \]

**H Likelihood function of the commodity future contract**

In this section we derive the likelihood function of the commodity future contract. We can state that the non observable variable \( \delta_{t_i} \) is related to the observable variable \( F (t_i, T_3) \) by a reciprocal bijective transformation denoted \( M (\bullet; \kappa, \theta, \sigma_\delta) \), such that, for any \( i \in (1, 2, ..., N) \), we have

\[ F (t_i, T_3) = M (\delta_{t_i}; \kappa, \theta, \sigma_\delta) \quad \text{and} \quad \delta_{t_i} = M^{-1} (F (t_i, T_3); \kappa, \theta, \sigma_\delta) , \]

where \( M (\bullet; \kappa, \theta, \sigma_\delta) \) is defined by the equation (11), and \( M^{-1} (\bullet; \kappa, \theta, \sigma_\delta) \) is given by the following relation:
\[ \delta_{t_i} = \left( \frac{a}{1 - e^{-a(T_3 - t_i)}} \right) + \ln \left( \frac{S_{t_i}}{F(t_i, T_3)} \right) + \int_{t_i}^{T_3} f(0, u) du + \frac{\sigma_f^2}{\delta_f^2} (T_3 - t_i) \left\{ -m + \frac{\sigma_S^2}{a^2} - \frac{\sigma_S \sigma_f}{a} + \frac{\sigma_f^2}{2} + \rho_{13} \sigma_f \left( \sigma_S - \frac{\sigma_f}{a} \right) \right\} (T_3 - t_i) \] 

Thus, according to the data transformation method, we can write

\[ L (F(t_i, T_3) : i \in (1, 2, ..., N); \kappa, \theta, \sigma_\delta) = L (\delta_{t_i} : i \in (1, 2, ..., N); \kappa, \theta, \sigma_\delta) \times |J|, \]

where \( J \) is the Jacobian of the inverse transformation

\[ J = \begin{vmatrix} \frac{\partial M^{-1}(F(t_1, T_3); \kappa, \theta, \sigma_\delta)}{\partial F(t_1, T_3)} & \cdots & \frac{\partial M^{-1}(F(t_1, T_3); \kappa, \theta, \sigma_\delta)}{\partial F(T_{N}, T_3)} \\ \vdots & \ddots & \vdots \\ \frac{\partial M^{-1}(F(t_N, T_3); \kappa, \theta, \sigma_\delta)}{\partial F(t_1, T_3)} & \cdots & \frac{\partial M^{-1}(F(t_N, T_3); \kappa, \theta, \sigma_\delta)}{\partial F(T_{N}, T_3)} \end{vmatrix}. \]

Using the expression of \( M^{-1}(\bullet; \kappa, \theta, \sigma_\delta) \) given by the equation (26), we get

\[ J = \prod_{i=1}^{N} \left( \frac{1 - e^{-a(T_3 - t_i)}}{1 - e^{-a(T_3 - t_i)}} F(t_i, T_3) \right). \]

Thus the likelihood function based on the sample of observed commodity future contract values is

\[ L (F(t_i, T_3) : i \in (1, 2, ..., N); \kappa, \theta, \sigma_\delta) = \left( \frac{1}{\sqrt{2\pi}} \right)^{N-1} \times \left( \frac{\sqrt{2\kappa}}{\sigma_\delta \sqrt{1 - e^{-2\kappa \Delta_t}}} \right)^{N-1} \times \prod_{i=2}^{N} \exp \left\{ -\kappa \left( \delta_{t_i} - \hat{\delta}_{t_{i-1}} e^{-\kappa \Delta_t} - \theta (1 - e^{-\kappa \Delta_t}) \right) \right\} \times \prod_{i=1}^{N} \left\{ \frac{-a}{(1 - e^{-a(T_3 - t_i)}) F(t_i, T_3)} \right\}, \]

such that \( \hat{\delta}_{t_i} \) is given by the equation (26)
I Proof of the geometric Brownian motion for gold price

As done in Gibson and Schwartz (1990), we analyzed the time series properties of the gold spot price in order to examine if the lognormal distribution is supported by the data. To do so, we collected 10 years of daily gold price data covering the period from February 1992 to February 2002 and regressed the logarithm of the price ratios on their lagged values

\[
\ln \left( \frac{S_{t+1}}{S_t} \right) = a + b \ln \left( \frac{S_t}{S_{t-1}} \right) + \xi_t.
\]

The results reported in Table 6 show that the constant and the coefficient \( b \) are not significantly different from zero, which means that the gold price seems to follow a random walk. In fact, the logarithm of the price ratio is normally distributed proving thus the use of a geometrical Brownian motion to modelize the gold spot price.

<table>
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<tr>
<th>Estimated coefficient</th>
<th>Standard errors</th>
<th>T-ratio</th>
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<td>0.00013</td>
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<tr>
<td>( b )</td>
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<td>0.01958</td>
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References


